Perturbation of Ornstein-Uhlenbeck stationary distributions: expansion and simulation

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Abstract

We consider a multidimensional stochastic differential equation Y written as a drift-perturbation of an ergodic Ornstein-Uhlenbeck process X. Under the condition of time-reversibility of X, we derive a first and second order expansion of the stationary distribution μ^Y of Y in terms of X. Error estimates are established. These approximations are then turned into a simulation scheme for sampling approximately according to μ^Y. Numerical experiments support the theoretical error estimates.

`Keywords:` ergodic diffusion, invariant measure, time-reversibility, perturbation method, asymptotic expansion

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1 Introduction

The problem. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space supporting a q-dimensional Brownian motion W, with $q \geq 1$ and the usual assumptions on the filtration. Consider the $\mathbb{R}^d$-valued solution $(Y_t)_{t\geq 0}$ of the stochastic differential equation

$$\mathrm{d}Y_t = \Big[- AY_t + \beta(Y_t)\Big]\mathrm{d}t + \Sigma \mathrm{d}W_t, \quad Y_0 \text{ independent of } W, \tag{1.1}$$

where A is $d \times d$-matrix, Σ is $d \times q$-matrix and $\beta : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a measurable function, quantities that satisfy assumptions specified later. We assume that Y has a unique invariant measure μ^Y, and we are concerned by the approximation

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and simulation of μ^Y using an instrumental process X and its invariant distribution μ^X (assuming so far it exists). Namely, X is the solution of the linear SDE (generalized Ornstein-Uhlenbeck equation)

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + \Sigma\mathrm{d}W_t, \quad X_0 \text{ independent of } W. \tag{1.2}$$

While μ^Y is hardly tractable and not easily simulable, μ^X is rather explicit (Gaussian distribution) and can be advantageously used to approximate μ^Y: our work provides a numerical scheme to achieve this purpose. Our main results (Theorems 1 and 2) state that

$$\mu^Y(\mathrm{d}x) \approx (1 + c_k(x))\, \mu^X(\mathrm{d}x) \tag{1.3}$$

with some explicit correction terms c_k that depend of the approximation order k. Our analysis is supported by theoretical error estimates and numerical experiments. Essentially we consider that Y is a perturbation of X, and our analysis may be justified by β small in a suitable sense.

Background results. Our strategy is based on stochastic approximation techniques, in the spirit of the proxy expansion of [GM14]. This is quite different from the literature on small noise expansion [FS86][FW98], or small time asymptotics [Wat87][KP10], or even multiscale asymptotics of [FPSK11]. To the best of our knowledge, this is the first time that the problem of approximating invariant distribution is tackled, which gives raise to new challenges compared to the previous works.

In this paper, we will only derive a limited number of correction terms in (1.3) (first and second order approximation), but in principle any order could be achieved. The main term is the stationary distribution of the OU process X. In practice, the first and second order correction terms provide an excellent accuracy. The final second order approximation takes the form

$$\mu^Y(\mathrm{d}x) \approx \mathbb{E}\left[\xi_2(\tau_1, \tau_2, X^x_{\tau_1}, X^x_{\tau_2}, X^x_{\tau_1+\tau_2}, x)\right] \mu^X(\mathrm{d}x)$$

where $\xi_2(\cdot)$ is explicit, X^x stands for the OU process starting from x and (τ_1, τ_2) are independent exponentially-distributed random variables. The above representation is very convenient to sample according to μ^Y, as we will see later.

Approximation strategy. It consists in defining a suitable interpolation between the distributions of X and Y, and performing expansion along this interpolation. Actually, this is done for the marginal of the processes at time $T \to +\infty$. The crucial part lies in the derivation of explicit correction terms. As a difference with strong approximation techniques like in [Wat87][FW98][KP10], we have to control in the weak sense X_T and Y_T as $T \to +\infty$, since strong estimates explode in large time. To achieve this weak convergence analysis, we use in an essential manner the reversibility of the process X to pull back the computations in large time to small time, and derive in this way explicit representations. Furthermore, when taking $T \to +\infty$ we show that expansion errors can be well controlled.

Outline of the paper. We first give notations used throughout the paper. Section 2 gathers preliminary properties on X and Y. In particular we provide necessary and sufficient conditions under which X is a reversible process; these results are interesting on their own, besides they will be crucially used in the subsequent expansion analysis. In Section 3, we state our main approximation results and analyse the errors. In Section 4, we investigate how to turn the expansion results into simulation algorithms of μ^Y and we perform some numerical experiments.

Notations and basic definitions. The following are used frequently throughout this paper.

Vector. $|x|$ is the Euclidean norm of a vector $x \in \mathbb{R}^d$. Its sup-norm is denoted by $|x|_\infty$.

Its i-th coordinate is x_i, or $x_{i,t}$ if $x = x_t$ depends on time.

Matrix. If $A = (A_{i,j})$ is a square matrix, $A^\top$ denotes its transpose, $\mathrm{Tr}(A)$ its trace and $\det(A)$ its determinant. $\lambda(A)$ denotes the spectrum of A.

Id is the $d \times d$-identity matrix.

For a symmetric matrix A, $\lambda_{\max,A}$ and $\lambda_{\min,A}$ stand for the maximum and minimum eigenvalues of A.

Definition 1. [Kha02, p. 135] *A square matrix H is called a* Hurwitz matrix *if its eigenvalues have negative real parts, i.e. its spectrum $\lambda(H)$ is included in $\{\lambda \in \mathbb{C} : \mathcal{R}\mathrm{e}(\lambda) < 0\}$.*

$\|A\|$ is the matrix norm of A subordinated to the Euclidean norm.

$\mathrm{vec}(A)$ is the vectorizing operator applied to A, it stacks the columns of A into a vector.

The Kronecker product $A \otimes B$ of matrix $A = (A_{ij})_{1\le i\le m, 1\le j\le n}$ and $B = (B_{ij})_{1\le i\le k, 1\le j\le l}$ is defined as

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{bmatrix},$$

where $A \otimes B$ is a $mk \times nl$ matrix.

We also recall that if the matrices M_1, M_2 and M_3 are conformable for matrix multiplication then

$$\mathrm{vec}\,(M_1 M_2 M_3) = \left(M_3^\top \otimes M_1\right) \mathrm{vec}\,(M_2)\,. \tag{1.4}$$

If M_1 and M_2 are two matrices of the same size, then

$$\mathrm{Tr}\left(M_1^\top M_2\right) = \left(\mathrm{vec}\,(M_1)\right)^\top \mathrm{vec}\,(M_2)\,.$$

Functions. For a vector (resp. matrix) valued function φ, $|\varphi|_\infty$ stands for $\sup_{t,x} |\varphi(t,x)|$ (resp. $\sup_{t,x} \|\varphi(t,x)\|$).

For a smooth function $\varphi(w)$, $\partial_{w_i}\varphi(w)$ stands for the partial derivative of φ with respect to w_i.

If $\varphi : \mathbb{R}^d \to \mathbb{R}^q$, then $\nabla\varphi(x)$ is defined as the $q \times d$-matrix

$$\nabla\varphi(x) := \left[\partial_{x_1}\varphi(x), \cdots, \partial_{x_d}\varphi(x)\right].$$

For a smooth drift function $\beta : \mathbb{R}^d \mapsto \mathbb{R}^d$, we set $\beta^{(1)}(x) := \nabla\beta(x)$, $\beta^{(2)}(x) := \nabla \operatorname{vec}(\beta^{(1)})(x)$ and $\beta^{(3)}(x) := \nabla \operatorname{vec}(\beta^{(2)})(x)$.

For a smooth test function $h : \mathbb{R}^d \mapsto \mathbb{R}$, we set $h^{(1)}(x) := \nabla h(x)$, $h^{(2)}(x) := (\partial_{x_i,x_j} h(x))_{i,j}$ and $h^{(3)}(x) := \nabla \operatorname{vec}(h^{(2)})(x)$.

The divergence of $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is $\operatorname{div}(\varphi) := \operatorname{Tr}(\nabla\varphi)$.

Generic constants. We shall denote by c all constants which depend only on universal constants and the dimension d.

2 Preliminary results

2.1 Exponential of a Hurwitz matrix

We will repeatedly use the large time behavior of e^{Ht} as $t \to +\infty$, for a given square matrix H. Consider its Jordan matrix canonical decomposition [FF07, Chapter 1, Section 7], $H = T_H J_H T_H^{-1}$, for which T_H comprises the generalized eigenvectors of H and J_H is block diagonal, i.e.

$$J_H = \begin{pmatrix} J_{k_{1,H}}(\lambda_{1,H}) & 0 & 0 & \dots & 0 \\ 0 & J_{k_{2,H}}(\lambda_{2,H}) & 0 & \dots & 0 \\ 0 & 0 & J_{k_{3,H}}(\lambda_{3,H}) & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & J_{k_{m,H}}(\lambda_{m,H}) \end{pmatrix}$$

and each block has the form

$$J_{k_{i,H}}(\lambda_{i,H}) = \begin{pmatrix} \lambda_{i,H} & 1 & 0 & \dots & 0 \\ 0 & \lambda_{i,H} & 1 & \dots & 0 \\ 0 & 0 & \lambda_{i,H} & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \dots & \lambda_{i,H} \end{pmatrix} (k_{i,H} \times k_{i,H})$$

where $k_{i,H}$ is the multiplicity of $\lambda_{i,H}$.

Then, we can write $\exp(Ht) = T_H \exp(J_H t) T_H^{-1}$. Since J_H is block diagonal, so is $\exp(J_H t)$; in particular, for each submatrix of the form $J_{k_{i,H}}(\lambda_{i,H})$ on the block diagonal of J_H, $\exp(J_H t)$ will contain the diagonal block

$$\exp(J_{k_{i,H}}(\lambda_{i,H})t) = \exp(\lambda_{i,H}t) \begin{pmatrix} 1 & t & \frac{1}{2}t^2 & \dots & \frac{1}{(k_{i,H}-1)!}t^{k_{i,H}-1} \\ 0 & 1 & t & \dots & \frac{1}{(k_{i,H}-2)!}t^{k_{i,H}-2} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & t \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} (k_{i,H} \times k_{i,H}).$$

Then, it becomes clear that if H is a Hurwitz matrix, e^{Ht} converges to 0, with some precise exponential rate. We have proved the following result.

Lemma 1. *If H is a Hurwitz matrix, then for any $\lambda_H \in (0, -\max_{\lambda \in \lambda(H)} \mathcal{R}e(\lambda))$ there exist $c_H > 0$ and $c_{\mathrm{vec}\,(H)} > 0$ such that*

$$\left\| e^{Ht} \right\| \leq c_H e^{-\lambda_H t} \quad \textit{and} \quad \left| \mathrm{vec}\left(e^{Ht}\right) \right| \leq c_{\mathrm{vec}\,(H)} e^{-\lambda_H t} \qquad \forall t \geq 0.$$

2.2 The generalized Ornstein-Uhlenbeck process

We collect in this paragraph the main results on X. It will serve as a basis for all this work. The two following statements are standard, see for instance [KS91, Section 5.6].

Proposition 1. *Assume that the random variable X_0 is independent of W and is square integrable. Then, there exists a unique square integrable solution to* (1.2)*, represented as*

$$X_t = e^{-At} \left[X_0 + \int_0^t e^{Ar} \Sigma \mathrm{d}W_r \right]. \tag{2.1}$$

Its mean $m_t := \mathbb{E}(X_t)$ and covariance $V_{t,s} := \mathbb{E}(X_t X_s^\top)$ are given by

$$m_t := e^{-At}\mathbb{E}(X_0), \tag{2.2}$$

$$V_{t,s} := e^{-At} \left(\mathbb{E}(X_0 X_0^\top) + \int_0^{t \wedge s} e^{Ar} \Sigma \Sigma^\top e^{A^\top r} \mathrm{d}r \right) e^{-A^\top s} \tag{2.3}$$

for any $s \geq 0$ and $t \geq 0$. Additionally, for any $t \geq s \geq 0$, the distribution of X_t conditioned on X_s is Gaussian

$$\mathcal{N}\left(e^{-A(t-s)} X_s, \int_0^{t-s} e^{-Ar} \Sigma \Sigma^\top e^{-A^\top r} \mathrm{d}r \right). \tag{2.4}$$

Proof. (2.1) is a direct consequence of the Itô formula. Identities (2.2)-(2.3)-(2.4) are straightforward. □

Proposition 2. *Assume that $-A$ is a Hurwitz matrix. Then, X has a unique stationary distribution μ^X which is Gaussian with mean 0 and covariance*

$$V_\infty := \int_0^{+\infty} e^{-Ar} \Sigma \Sigma^\top e^{-A^\top r} \mathrm{d}r. \tag{2.5}$$

Under the stationary distribution, we have

$$V_{t,s} = e^{-A(t-s)} V_\infty \tag{2.6}$$

for any $t \geq s \geq 0$.

Proof. Owing to Lemma 1, both matrices e^{-At} and $e^{-A^\top t}$ converge to 0 exponentially fast as $t \to +\infty$. Then, the stochastic integral $e^{-At}\int_0^t e^{Ar}\Sigma \mathrm{d}W_r$ is Gaussian, centered, with covariance $\int_0^t e^{-As}\Sigma\Sigma^\top e^{-A^\top s} \to V_\infty$ as $t \to +\infty$: thus, the related Wiener stochastic integral converges weakly to μ^X. Since $e^{-At}X_0$ converges almost surely to 0 and in view of (2.1), we have proved the weak convergence of X to μ^X. If X_0 has the Gaussian distribution μ^X, then X is a Gaussian process, with mean 0 (see (2.2)): the covariance matrix of X_t is (see (2.3))

$$V_{t,t} = e^{-At}\left(V_\infty + \int_0^t e^{Ar}\Sigma\Sigma^\top e^{A^\top r}\mathrm{d}r\right)e^{-A^\top t} = V_\infty. \tag{2.7}$$

From the above and (2.3), we easily deduce (2.6). □

In our subsequent approximation approach (Section 3), we make use of the property of time-reversibility of X, i.e. for any non-negative times t and s, under the stationary distribution the processes $(X_r)_{s\leq r\leq t}$ and $(X_{s+t-r})_{s\leq r\leq t}$ have the same distribution. In the case of discrete-time Gaussian linear processes, this time-reversibility question is investigated in [TZ05]. Here in our continuous-time framework, we establish a rather explicit equivalence criterion with a specific proof. To the best of our knowledge, this statement is new.

Proposition 3. *Assume that the assumptions of Proposition 2 hold and that X is considered under the stationary distribution μ^X. Then X is reversible if and only if $A\Sigma\Sigma^\top$ is a symmetric matrix, i.e.*

$$A\Sigma\Sigma^\top = \Sigma\Sigma^\top A^\top. \tag{2.8}$$

Furthermore, we have in this case

$$V_\infty = \frac{1}{2}A^{-1}\Sigma\Sigma^\top. \tag{2.9}$$

Proof. Note that (2.9) directly follows from (2.5) and (2.8).

Because of the Gaussian properties of X, time-reversibility is equivalent to $V_{t,s} = V_{s,t}$ for any $t \geq s \geq 0$. Observe that $V_{s,t} = V_{t,s}^\top$.
Proof of $\Longleftarrow$: we first prove by induction that $A^k\Sigma\Sigma^\top = \Sigma\Sigma^\top[A^\top]^k$ for any $k \geq 1$. This is true for $k = 1$ because of (2.8); assuming the property for k, then $A^{k+1}\Sigma\Sigma^\top = A[A^k\Sigma\Sigma^\top] = A[\Sigma\Sigma^\top[A^\top]^k] = \Sigma\Sigma^\top A^\top[A^\top]^k = \Sigma\Sigma^\top[A^\top]^{k+1}$, therefore the announced property.

Second, by writing the matrix exponential as a series, it readily follows that $e^{-At}\Sigma\Sigma^\top = \Sigma\Sigma^\top e^{-A^\top t}$ for any $t \geq 0$. We finally deduce

$$V_{t,s} \overset{(2.6)}{=} e^{-A(t-s)}V_\infty \overset{(2.5)}{=} \int_0^{+\infty} e^{-Ar}e^{-A(t-s)}\Sigma\Sigma^\top e^{-A^\top r}\mathrm{d}r$$

$$\overset{(2.8)}{=} \int_0^{+\infty} e^{-Ar}\Sigma\Sigma^\top e^{-A^\top(t-s)}e^{-A^\top r}\mathrm{d}r$$
$$= V_\infty e^{-A^\top(t-s)} = V_{s,t},$$

therefore the time-reversibility of X.
Proof of $\Longrightarrow$: assume $V_{t,s} = V_{s,t}$ for any $t \geq s \geq 0$. For such t and s, write $V_{t,s}$ in a form similar to (2.7), i.e.

$$V_{t,s} = e^{-At}\left(V_\infty + \int_0^s e^{Ar}\Sigma\Sigma^\top e^{A^\top r}\mathrm{d}r\right)e^{-A^\top s}.$$

A straightforward differentiation w.r.t. s and t ($t > s$) gives

$$\begin{aligned} \partial_t V_{t,s} &= -AV_{t,s}, \qquad \partial_s V_{t,s} = -V_{t,s}A^\top + e^{A(s-t)}\Sigma\Sigma^\top, \\ \partial^2_{s,t}V_{t,s} &= AV_{t,s}A^\top - Ae^{A(s-t)}\Sigma\Sigma^\top. \end{aligned} \tag{2.10}$$

Owing to $V_{s,t} = V_{t,s}^\top$, we deduce $\partial^2_{s,t}V_{s,t} = (AV_{t,s}A^\top)^\top - (Ae^{A(s-t)}\Sigma\Sigma^\top)^\top$. Now, invoking the time-reversibility and identifying the previous equality with (2.10), we obtain $Ae^{-Ar}\Sigma\Sigma^\top = \Sigma\Sigma^\top e^{-A^\top r}A^\top$ for any $r = t - s > 0$. By taking the limit as $r \downarrow 0$, we obtain the advertised equality (2.8). □

Examples 1. *Let us consider the assumptions of Proposition 2.*

i) If A is symmetric and A and $\Sigma\Sigma^\top$ commute, then reversibility holds.

ii) In the case $\Sigma\Sigma^\top = c\,\mathrm{Id}$ for some $c > 0$, reversibility is equivalent to symmetry of A.

iii) The symmetry of A is not sufficient for time-reversibility: for instance, take $A = \mathrm{diag}(1,2)$ and $\Sigma\Sigma^\top = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ for $\rho \neq 0$. Indeed $A\Sigma\Sigma^\top = \begin{pmatrix} 1 & \rho \\ 2\rho & 2 \end{pmatrix}$ is not symmetric, therefore X is not time-reversible.

iv) There are many situations of reversibility for which A is not symmetric: take $\Sigma\Sigma^\top$ as before and set $A = \begin{pmatrix} 1 & \rho(\lambda - 1) \\ 0 & \lambda \end{pmatrix}$ for some $\lambda \in (0, +\infty)\backslash\{1\}$. A is not symmetric but $A\Sigma\Sigma^\top$ is, there reversibility holds.

Extension to non-centered Ornstein-Uhlenbeck process. For the sake of completeness, we briefly expose how to handle the case of a constant term in the drift coefficient of (1.1) and (1.2), by using a simple translation of the variables. Namely, consider

$$\mathrm{d}X^a_t = (-AX^a_t + a)\mathrm{d}t + \Sigma\mathrm{d}W_t, \quad X^a_0 \text{ independent of } W,$$

with $a \in \mathbb{R}^d$, and similarly for Y. Consider the assumptions of Proposition 2 and set $m = a\int_0^\infty e^{-Ar}\mathrm{d}r$. A similar analysis yields that X^a has a stationary Gaussian distribution $\mathcal{N}(m, V_\infty)$ (see [KS91, Eq. (6.10-6.11) p.355]): moreover, we have the relation $Am = a$ (see [KS91, Eq. (6.12) p.355]). Now, if we put $X_t = X_t^a - m$, we obtain $\mathrm{d}X_t = (-AX_t + (a - Am))\mathrm{d}t + \Sigma\mathrm{d}W_t = -AX_t\mathrm{d}t + \Sigma\mathrm{d}W_t$. It means that all subsequent results can be translated from the case $a = 0$ to the case $a \neq 0$, by shifting the space variable from m. In the error estimates of Theorems 1 and 2, we would have to replace β by $\beta - a$. Heuristically, for a given β, the parameter a should be chosen such that m is approximately equal to the mean of μ^Y. These heuristics are not captured in our subsequent quantitative analysis, however this centering property presumably brings further accuracy.

2.3 The perturbed Ornstein-Uhlenbeck process

Recall the process of interest Y defined by (1.1). We impose that β is a Lipschitz function, ensuring the existence and uniqueness of a strong solution (with square integrability) for any initial condition Y_0 independent of W and square integrable (see [KS91, Theorem 2.9 p.289]). Reinforcing the conditions lead to ergodic properties for Y, as stated in the following standard result.

Proposition 4. *Assume that $-A$ is a Hurwitz matrix, that $\Sigma\Sigma^\top$ is non-singular and that β is a bounded Lipschitz function. Then, Y has a unique stationary distribution μ^Y, which has a density with respect to the Lebesgue measure; thus we denote $\mu^Y(\mathrm{d}y) = \mu^Y(y)\mathrm{d}y$.*

Furthermore, for any continuous bounded function $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ and any $x \in \mathbb{R}^d$, we have

$$\lim_{t\to+\infty} \mathbb{E}[\varphi(Y_t)|Y_0 = x] = \int_{\mathbb{R}^d} \varphi(y)\mu^Y(\mathrm{d}y). \tag{2.11}$$

Proof. To prove the existence and uniqueness of a stationary distribution, we apply [Kha12, Theorem 4.1 and Corollary 4.4] by checking the validity of [Kha12, assumptions (B1)-(B2), p.107]. The related assumption (B1) is satisfied since $\Sigma\Sigma^\top$ is non-degenerate. To check (B2) (the mean recurrence time is locally uniform w.r.t. the starting point), we apply [Kha12, Theorem 3.9] with the Lyapunov function $V(x) = x \cdot P^\# x$ where $P^\#$ is a symmetric non-negative matrix to be specified hereafter. Denoting by L^Y the infinitesimal generator associated to Y, it reduces to check only that $L^Y V(x) \leq -1$ for $|x|$ large enough. Simple computations give

$$\begin{aligned} L^Y V(x) &= 2x^\top P^\#(-Ax + \beta(x)) + \mathrm{Tr}(P^\#\Sigma\Sigma^\top) \\ &= -x^\top(P^\# A + A^\top P^\#)x + 2x \cdot P^\#\beta(x) + \mathrm{Tr}(P^\#\Sigma\Sigma^\top) \\ &\leq -x^\top(P^\# A + A^\top P^\#)x + \frac{1}{2}|x|^2 + 2|P^\#\beta|_\infty^2 + \mathrm{Tr}(P^\#\Sigma\Sigma^\top). \end{aligned}$$

Now, for a given symmetric positive-definite matrix Q we consider solutions P to the Lyapunov equation

$$PA + A^\top P = Q.$$

By [Kha02, Theorem 4.6] and since $-A$ is Hurwitz, there is a unique solution P_Q in the class of symmetric positive-definite matrices. Denote by $P^\#$ the solution associated to $Q = \mathrm{Id}$. This choice ensures that the related Lyapunov function V satisfies $L^Y V(x) \leq -\frac{1}{2}|x|^2 + 2|P^{\#}\beta|_\infty^2 + \mathrm{Tr}(P^{\#}\Sigma\Sigma^\top)$, therefore the requested behavior for $|x|$ large.

The existence of of a density with respect to the Lebesgue measure is a direct consequence of [Kha12, Lemma 4.16].

Finally, [Kha12, Theorem 4.3] gives the convergence (2.11) and we are done.

□

3 Main approximation results

We start by listing the working assumptions we need, in order to ensure the validity of approximation.

(H-i) $-A$ is a Hurwitz matrix.

Choose $\lambda_{0,A} \in (0, \min_{\lambda \in \lambda(A)} \mathcal{R}e\,(\lambda))$: for such $\lambda_{0,A}$, define

$$A_0 := A - \lambda_{0,A}\mathrm{Id};$$

note that $-A_0$ is a Hurwitz matrix. Denote by $c_A > 0$ the constant such that $\left\|e^{-As}\right\| \leq c_A e^{-\lambda_{0,A}s}$, $\forall s \geq 0$ (see Lemma 1).

(H-ii) $A\Sigma\Sigma^\top$ is symmetric.

(H-iii) $\Sigma\Sigma^\top$ is non-singular.

(H-iv) β is a $d \times 1$ vector and all its derivatives up to third order are bounded.

(H-v) h is a continuous bounded function and its derivatives up to third order exist and are bounded.

(H-vi) $\lambda_{0,A} > c_A|\beta^{(1)}|_\infty$, where $\lambda_{0,A}, c_A > 0$ are the constants from **(H-i)**. We set

$$c_{(3.1)} := \frac{c_A}{\lambda_{0,A} - c_A|\beta^{(1)}|_\infty}. \tag{3.1}$$

3.1 Strategy

For the sake of pedagogy, we outline the main arguments for the first-order approximation only. The full justification is given in Subsection 3.3. We aim at approximating

$$\int_{\mathbb{R}^d} h(y)\mu^Y(y)\mathrm{d}y - \int_{\mathbb{R}^d} h(x)\mu^X(x)\mathrm{d}x$$

for some test functions h in a certain class (assumption **(H-v)**). To study the approximation of μ^Y by μ^X, we use the solution denoted by $\tilde{Y}$ starting from μ^X (independent of W):

$$\mathrm{d}\tilde{Y}_t = \left(-A\tilde{Y}_t + \beta(\tilde{Y}_t)\right)\mathrm{d}t + \Sigma\mathrm{d}W_t, \qquad \tilde{Y}_0 \overset{d}{=} \mathcal{N}(0, V_\infty) = \mu^X.$$

By the sake of symmetry, we also denote by $\tilde{X}$ the process solution without the above β-term, starting from μ^X.

Finite-time approximation. Under the standing assumption **(H-i)**, recall that V_∞ is properly defined in (2.5). For a given $T > 0$, consider furthermore that

$$\mathbb{E}_x\left[h(\tilde{Y}_T)\right] := \mathbb{E}\left[h(\tilde{Y}_T) \mid \tilde{Y}_0 = x\right],$$
$$\mathbb{E}_{\mu^X}\left[h(\tilde{Y}_T)\right] := \int_{\mathbb{R}^d} \mathbb{E}_x\left[h(\tilde{Y}_T)\right] \mu^X(x)\mathrm{d}x.$$

First, we get the following lemma.

Lemma 2. *Assume* **(H-i)**-**(H-ii)**-**(H-iii)**-**(H-iv)**. *For any continuous bounded function $h : \mathbb{R}^d \to \mathbb{R}$, we have*

$$\lim_{T\to+\infty} \mathbb{E}_{\mu^X}\left[h(\tilde{Y}_T)\right] = \int_{\mathbb{R}^d} h(y)\mu^Y(y)\mathrm{d}y, \tag{3.2}$$
$$\lim_{T\to+\infty} \mathbb{E}_{\mu^X}\left[h(\tilde{X}_T)\right] = \int_{\mathbb{R}^d} h(x)\mu^X(x)\mathrm{d}x. \tag{3.3}$$

Parameterization. Then, we introduce the d-dimensional parameterized process

$$\mathrm{d}\tilde{X}^\epsilon_t = \left(-A\tilde{X}^\epsilon_t + \epsilon\beta(\tilde{X}^\epsilon_t)\right)\mathrm{d}t + \Sigma\mathrm{d}W_t, \qquad \tilde{X}^\epsilon_0 \overset{d}{=} \mu^X \tag{3.4}$$

for $\epsilon \in [0, 1]$ and we define

$$\tilde{X}^\epsilon_{i,t} = \frac{\partial^i \tilde{X}^\epsilon_t}{\partial \epsilon^i}, \qquad \tilde{X}_{i,t} = \left.\frac{\partial^i \tilde{X}^\epsilon_t}{\partial \epsilon^i}\right|_{\epsilon=0}.$$

These derivatives up to the second order are well defined by an application of [Kun97]. A simple computation shows that $\tilde{X}_{0,t} = \tilde{X}_t$ and

$$\mathrm{d}\tilde{X}_{1,t} = \left(-A\tilde{X}_{1,t} + \beta(\tilde{X}_t)\right)\mathrm{d}t, \qquad \tilde{X}_{1,0} = 0. \tag{3.5}$$

With this d-dimensional parameterization, we can derive a decomposition of $\mathbb{E}_{\mu^X}\left[h(\tilde{Y}_T)\right]$.

Decomposition. For a continuous bounded function h, with bounded derivatives, we expand $\mathbb{E}_{\mu^X}\left[h(\tilde{Y}_T)\right]$ to the first order. Applying the Taylor expansion to $\mathbb{E}_{\mu^X}\left[h(\tilde{X}^\epsilon_T)\right]$ for $\epsilon = 1$ around $\epsilon = 0$, we have

$$\mathbb{E}_{\mu^X}\left[h(\tilde{Y}_T)\right] = \mathbb{E}_{\mu^X}\left[h(\tilde{X}_T)\right] + C_1(T) + \mathrm{Error}_1(T) \tag{3.6}$$

where

$$\begin{aligned} C_1(T) &:= \mathbb{E}_{\mu^X}\left[h^{(1)}(\tilde{X}_T)\tilde{X}_{1,T}\right], \\ \mathrm{Error}_1(T) &:= \int_0^1 (1-u)\,\mathbb{E}_{\mu^X}\Big[(\tilde{X}^u_{1,T})^\top h^{(2)}(\tilde{X}^u_{0,T})(\tilde{X}^u_{1,T}) \\ &\qquad + h^{(1)}(\tilde{X}^u_{0,T})\tilde{X}^u_{2,T}\Big]\mathrm{d}u. \end{aligned} \tag{3.7}$$

Taking limits as $T \to +\infty$. Owing to Lemma 2, the first two terms of (3.6) converge respectively to

$$\int_{\mathbb{R}^d} h(y)\mu^Y(y)\mathrm{d}y \qquad \text{and} \qquad \int_{\mathbb{R}^d} h(y)\mu^X(y)\mathrm{d}y.$$

Besides, we will also show that the first correction term converges (see Equation (3.24)):

$$\lim_{T\to+\infty} C_1(T) = \int_{\mathbb{R}^d} h(x)c_1(x)\mu^X(x)\mathrm{d}x$$

where $c_1(x)$ is given in Theorem 1. Here, we use *Time-Reversibility of X* to obtain Equation (3.21), which allows us to derive the above limit. Moreover, we also prove that the error term, $\mathrm{Error}_1(T)$, is uniformly bounded (when $T \to +\infty$) and controlled like Estimate (3.11) by assuming Assumption **(H-vi)**. Therefore, this shows the approximate stationary distribution of μ^Y to be

$$\mu^{Y,1}(\mathrm{d}x) = (1 + c_1(x))\mu^X(\mathrm{d}x),$$

with an error bound for smooth test functions h.

3.2 First and second order expansions

We now state the main results obtained by using the above approximation strategy.

Theorem 1 (First-order approximation)**.** *If we assume* **(H-i)** *to* **(H-vi)***, then we have*

$$\int_{\mathbb{R}^d} h(y)\mu^Y(y)\mathrm{d}y = \int_{\mathbb{R}^d} h(x)\left(1 + c_1(x)\right)\mu^X(x)\mathrm{d}x + \mathrm{Error}_1, \tag{3.8}$$

where

$$c_1(x) = \mathbb{E}_x\left[\frac{x^\top V_\infty^{-1} e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1})}{\lambda_{0,A}} - \frac{\mathrm{Tr}\left(e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A\tau_1}\right)}{\lambda_{0,A}}\right] \tag{3.9}$$

and $\tau_1 \stackrel{d}{=} Exp\left(\lambda_{0,A}\right)$ is independent of the Brownian motion W. Additionally, we have

$$|c_1(x)| \le \frac{c_A}{\lambda_{0,A}}|x|\left\|V_\infty^{-1}\right\||\beta|_\infty + \frac{dc_A^2|\beta^{(1)}|_\infty}{2\lambda_{0,A}}, \tag{3.10}$$

$$|\mathrm{Error}_1| \le c\left(c_{(3.1)}^2|\beta|_\infty^2|h^{(2)}|_\infty \right.$$
$$\left. + \left(c_{(3.1)}^2|\beta^{(1)}|_\infty|\beta|_\infty + c_{(3.1)}^3|\beta^{(2)}|_\infty|\beta|_\infty^2\right)|h^{(1)}|_\infty\right). \tag{3.11}$$

As a consequence, the stationary distribution of Y is approximated by

$$\mu^{Y,1}(\mathrm{d}x) := \left(1 + c_1(x)\right)\mu^X(\mathrm{d}x). \tag{3.12}$$

We refer to (3.8) as a first-order approximation since the error term is quadratic in β and its derivatives. For the purpose of simulation in Section 4, we shall compute an explicit upper bound on c_1 as we do in (3.10).

Observe that $\mu^{Y,1}(x)\mathrm{d}x$ is a finite measure on $\mathbb{R}^d$ (owing to (3.10)), its mass equals 1 (take $h=1$ in (3.8)) but it is not necessarily a density (even for small β). This is discussed later in Section 4.

Theorem 2 (Second-order approximation)**.** *If we assume* **(H-i)** *to* **(H-vi)**, *then we have*

$$\int_{\mathbb{R}^d} h(y)\mu^Y(y)\mathrm{d}y = \int_{\mathbb{R}^d} h(x)\left(1+c_1(x)+c_{21}(x)+c_{22}(x)\right)\mu^X(x)\mathrm{d}x \\ + \mathrm{Error}_2, \tag{3.13}$$

where

$$\begin{aligned} c_{21}(x) = \mathbb{E}_x\Bigg[& \frac{x^\top V_\infty^{-1} e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A_0\tau_2}\beta(\tilde{X}_{\tau_1+\tau_2})}{\lambda_{0,A}^2} \\ & -\frac{1}{\lambda_{0,A}^2}\operatorname{Tr}\Bigg(\left\{\left(\beta^\top(\tilde{X}_{\tau_1+\tau_2})e^{-A_0^\top\tau_2}\right)\otimes e^{-A_0\tau_1}\right\}\beta^{(2)}(\tilde{X}_{\tau_1})e^{-A\tau_1} \\ & \qquad + e^{-A\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A\tau_2}\beta^{(1)}(\tilde{X}_{\tau_1+\tau_2})e^{-A_0(\tau_1+\tau_2)}\Bigg)\Bigg] \end{aligned}$$

and

$$\begin{aligned} c_{22}(x) = \mathbb{E}_x\Bigg[& \Big\{-x^\top V_\infty^{-1}e^{-A\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A_0(\tau_1+\tau_2)}\beta(\tilde{X}_{\tau_2}) \\ & + \operatorname{Tr}\left(\left\{\left(\beta^\top(\tilde{X}_{\tau_2})e^{-A_0^\top(\tau_1+\tau_2)}\right)\otimes e^{-A\tau_1}\right\}\beta^{(2)}(\tilde{X}_{\tau_1})e^{-A\tau_1}\right) \\ & + \operatorname{Tr}\left(e^{-A\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A_0(\tau_1+\tau_2)}\beta^{(1)}(\tilde{X}_{\tau_2})e^{-A\tau_2}\right) \\ & + \beta^\top(\tilde{X}_{\tau_2})e^{-A_0^\top\tau_2}V_\infty^{-1}xx^\top V_\infty^{-1}e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1}) \\ & - x^\top V_\infty^{-1}e^{-A\tau_2}\beta^{(1)}(\tilde{X}_{\tau_2})e^{-A_0(\tau_1+\tau_2)}\beta(\tilde{X}_{\tau_1}) \\ & - \beta^\top(\tilde{X}_{\tau_2})e^{-A_0^\top\tau_2}V_\infty^{-1}e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1}) \\ & - x^\top V_\infty^{-1}e^{-A_0\tau_2}\beta(\tilde{X}_{\tau_2})\operatorname{Tr}\left(e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A\tau_1}\right) \\ & - x^\top V_\infty^{-1}e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1})\operatorname{Tr}\left(e^{-A_0\tau_2}\beta^{(1)}(\tilde{X}_{\tau_2})e^{-A\tau_2}\right) \\ & + \left(\operatorname{vec}\left(e^{-2A^\top\tau_2}\right)\right)^\top\beta^{(2)}(\tilde{X}_{\tau_2})e^{-A_0(\tau_1+\tau_2)}\beta(\tilde{X}_{\tau_1}) \\ & + \operatorname{Tr}\left(e^{-A_0\tau_2}\beta^{(1)}(\tilde{X}_{\tau_2})e^{-A\tau_2}\right)\operatorname{Tr}\left(e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A\tau_1}\right)\Big\}\frac{\mathbb{1}_{\tau_1<\tau_2}}{\lambda_{0,A}^2}\Bigg] \end{aligned}$$

with $\tau_2 \overset{d}{=} Exp\left(\lambda_{0,A}\right)$ *independent of* τ_1 *and the Brownian motion* W*. Additionally, we have*

$$|c_{21}(x)| \le \frac{c_A^2\left\|V_\infty^{-1}\right\||\beta|_\infty|\beta^{(1)}|_\infty|x|}{\lambda_{0,A}^2} + \frac{dc_A^3|\beta|_\infty|\beta^{(2)}|_\infty}{2\lambda_{0,A}^2} + \frac{dc_A^3|\beta^{(1)}|_\infty^2}{4\lambda_{0,A}^2}$$

and

$$
\begin{aligned}
|c_{22}(x)| \leq \; & \frac{c_A|\beta^{(2)}|_\infty|\beta|_\infty}{4\lambda_{0,A}^2}\left(dc_A^2+\frac{c_{\mathrm{vec}(-2A^\top)}}{3}\right)+\frac{dc_A^3|\beta^{(1)}|_\infty^2}{8\lambda_{0,A}^2}\left(dc_A+1\right)\\
&+\frac{c_A^2\left\|V_\infty^{-1}\right\||\beta^{(1)}|_\infty|\beta|_\infty|x|}{2\lambda_{0,A}^2}\left(1+dc_A\right)\\
&+\frac{c_A^2\left\|V_\infty^{-1}\right\||\beta|_\infty^2}{2\lambda_{0,A}^2}\left(|x|^2\left\|V_\infty^{-1}\right\|+1\right).
\end{aligned}
$$

Moreover, the Error term is bounded by

$$
\begin{aligned}
&|\mathrm{Error}_2|\\
&\leq c\Bigg(|h^{(3)}|_\infty c_{(3.1)}^3|\beta|_\infty^3+|h^{(2)}|_\infty c_{(3.1)}^3|\beta|_\infty^2\left(|\beta^{(1)}|_\infty+c_{(3.1)}|\beta^{(2)}|_\infty|\beta|_\infty\right)\\
&\qquad +|h^{(1)}|_\infty c_{(3.1)}^3|\beta|_\infty\Bigg[\left(|\beta^{(1)}|_\infty+c_{(3.1)}|\beta^{(2)}|_\infty|\beta|_\infty\right)^2\\
&\qquad\qquad +|\beta|_\infty\left(|\beta^{(2)}|_\infty+c_{(3.1)}|\beta^{(3)}|_\infty|\beta|_\infty\right)\Bigg]\Bigg). \qquad (3.14)
\end{aligned}
$$

Consequently, the stationary distribution of Y is approximated by

$$
\mu^{Y,2}(\mathrm{d}x) := \left(1+c_1(x)+c_{21}(x)+c_{22}(x)\right)\mu^X(\mathrm{d}x). \qquad (3.15)
$$

Since the error term is cubic in β and its derivatives, (3.13) is a second-order approximation. Here again, $\mu^{Y,2}(x)\mathrm{d}x$ is a finite measure on $\mathbb{R}^d$ with mass equal to 1 but not necessarily a probability density.

3.3 Proof of Theorem 1

The proof is split into several steps.

Step 1) Preliminary estimates. The following result is instrumental in our analysis.

Proposition 5. *Let Z be the $\mathbb{R}^d$-valued solution to*

$$
dZ_s = \left(\left(-A+\alpha_s\right)Z_s+\gamma_s\right)\mathrm{d}s, \quad Z_0=0,
$$

where α and γ are two bounded measurable functions (resp. a $d\times d$ matrix and a vector in $\mathbb{R}^d$), and A is a $d\times d$ matrix function such that $\left\|e^{-As}\right\|\leq c_Ae^{-\lambda_{0,A}s}$ ($c_A>0, \lambda_{0,A}>0$). Then, if $\lambda_{0,A}>c_A|\alpha|_\infty$, we have

$$
\sup_{T\geq 0}|Z_T| \leq \frac{c_A|\gamma|_\infty}{\lambda_{0,A}-c_A|\alpha|_\infty}. \qquad (3.16)
$$

Moreover, if we assume that $B(t) := -A+\alpha_t$ commutes with $B(s)$ for every s and t, then Z writes

$$
Z_T = \int_0^T \exp\left(\int_s^T B(u)\mathrm{d}u\right)\gamma_s\mathrm{d}s. \qquad (3.17)
$$

Proof. The second statement is easily derived by writing (under the commutativity assumption on $(B(s) : s \geq 0)$)

$$e^{-\int_0^t B(u)\mathrm{d}u} Z_t = \int_0^t e^{-\int_0^s B(u)\mathrm{d}u} \gamma_s \mathrm{d}s,$$

then simplifying (still under the commutativity assumption).

The representation (3.17) can be an intermediate step in the derivation of (3.16). However, in our analysis, we need (3.16) without the restrictive commutativity assumption. We employ another approach using Gronwall inequality. Use (3.17) with the commutative family $B(s) = -A$ (by adjusting γ to $\gamma + \alpha Z$), hence

$$Z_t = \int_0^t e^{-A(t-s)} \gamma_s \mathrm{d}s + \int_0^t e^{-A(t-s)} \alpha_s Z_s \mathrm{d}s.$$

Then take the norm and apply Minkowski inequality: it follows

$$\begin{aligned}
|Z_t| &\leq \frac{c_A |\gamma|_\infty}{\lambda_{0,A}} + \int_0^t c_A |\alpha|_\infty e^{-\lambda_{0,A}(t-s)} |Z_s| \mathrm{d}s,\\
e^{\lambda_{0,A} t} |Z_t| &\leq \frac{c_A |\gamma|_\infty}{\lambda_{0,A}} e^{\lambda_{0,A} t} + \int_0^t c_A |\alpha|_\infty e^{\lambda_{0,A} s} |Z_s| \mathrm{d}s.
\end{aligned}$$

The Gronwall inequality yields

$$\begin{aligned}
e^{\lambda_{0,A} t} |Z_t| &\leq \frac{c_A |\gamma|_\infty}{\lambda_{0,A}} e^{\lambda_{0,A} t} + \int_0^t \frac{c_A^2 |\gamma|_\infty |\alpha|_\infty}{\lambda_{0,A}} e^{\lambda_{0,A} s} \exp\left(c_A |\alpha|_\infty (t-s)\right) \mathrm{d}s,\\
|Z_t| &\leq \frac{c_A |\gamma|_\infty}{\lambda_{0,A}} + \frac{c_A^2 |\gamma|_\infty |\alpha|_\infty}{\lambda_{0,A}} \int_0^t \exp\left(-\left(\lambda_{0,A} - c_A |\alpha|_\infty\right)(t-s)\right) \mathrm{d}s\\
&\leq \frac{c_A |\gamma|_\infty}{\lambda_{0,A} - c_A |\alpha|_\infty}.
\end{aligned}$$

□

The previous proposition allows us to directly control uniformly in time the process $\tilde{X}^\epsilon_{1,.}$, solution to

$$\mathrm{d}\tilde{X}^\epsilon_{1,t} = \left(-A\tilde{X}^\epsilon_{1,t} + \beta(\tilde{X}^\epsilon_{0,t}) + \epsilon \beta^{(1)}(\tilde{X}^\epsilon_{0,t}) \tilde{X}^\epsilon_{1,t}\right) \mathrm{d}t, \qquad \tilde{X}^\epsilon_{1,0} = 0.$$

Lemma 3. *Assume* **(H-i)**-**(H-iv)**-**(H-vi)**. *We have, for any* $\varepsilon \in [0, 1]$,

$$\sup_{T \geq 0} \left| \tilde{X}^\epsilon_{1,T} \right| \leq c_{(3.1)} |\beta|_\infty.$$

We proceed similarly for $\tilde{X}^\epsilon_{2,.}$, solution to

$$\begin{aligned}
\mathrm{d}\tilde{X}^\epsilon_{2,t} = \Big(&-A\tilde{X}^\epsilon_{2,t} + 2\beta^{(1)}(\tilde{X}^\epsilon_{0,t})\tilde{X}^\epsilon_{1,t} + \epsilon\beta^{(1)}(\tilde{X}^\epsilon_{0,t})\tilde{X}^\epsilon_{2,t}\\
&+ \epsilon \partial_\epsilon \left(\beta^{(1)}(\tilde{X}^\epsilon_{0,t})\right) \tilde{X}^\epsilon_{1,t}\Big) \mathrm{d}t, \qquad \tilde{X}^\epsilon_{2,0} = 0. \qquad (3.18)
\end{aligned}$$

Lemma 4. *Assume* **(H-i)**-**(H-iv)**-**(H-vi)**. *We have, for any* $\varepsilon \in [0,1]$,

$$\sup_{T\geq 0}\left|\tilde{X}^{\epsilon}_{2,T}\right| \leq c\left(c^2_{(3.1)}|\beta^{(1)}|_\infty|\beta|_\infty + c^3_{(3.1)}|\beta^{(2)}|_\infty|\beta|^2_\infty\right).$$

Proof. We observe that the matrix $\partial_\epsilon\left(\beta^{(1)}(\tilde{X}^{\epsilon}_{0,t})\right)$ has a norm bounded by $c|\beta^{(2)}|_\infty \sup_{T\geq 0}\left|\tilde{X}^{\epsilon}_{1,T}\right|$. Therefore, Proposition 5 combined with Lemma 3 yields

$$\sup_{T\geq 0}\left|\tilde{X}^{\epsilon}_{2,T}\right| \leq c_{(3.1)}\left(2|\beta^{(1)}|_\infty c_{(3.1)}|\beta|_\infty + c|\beta^{(2)}|_\infty\left(c_{(3.1)}|\beta|_\infty\right)^2\right).$$

□

Additionally, using Lemma 1, we easily prove the following.

Lemma 5. *Assume* **(H-i)**-**(H-iv)**. *We have*

$$\left|\mathrm{Tr}\left(e^{-A_0 r}\mathbb{E}_x\left[\beta^{(1)}(\tilde{X}_r)\right]e^{-Ar}\right)\right| \leq d\ c^2_A e^{-\lambda_{0,A}r}|\beta^{(1)}|_\infty.$$

Proof of Lemma 2. We prove a slightly more general statement regarding $\tilde{X}^{\epsilon}_T$ for any given $\epsilon \in [0,1]$. Denote by μ^{X^ε} its stationary distribution, which exists owing to Proposition 4. If $\tilde{X}^{\epsilon}_T$ follows the dynamics of Equation (3.4), then we have

$$\begin{aligned}\lim_{T\to+\infty}\mathbb{E}_{\mu^X}\left[h(\tilde{X}^{\epsilon}_T)\right] &= \lim_{T\to+\infty}\int_{\mathbb{R}^d}\mathbb{E}_x\left[h(\tilde{X}^{\epsilon}_T)\right]\mu^X(x)\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\mu^X(x)\mathrm{d}x\int_{\mathbb{R}^d}h(y)\mu^{X^\varepsilon}(y)\mathrm{d}y\end{aligned}$$

(by the dominated convergence theorem and Proposition 4)

$$= \int_{\mathbb{R}^d}h(y)\mu^{X^\varepsilon}(y)\mathrm{d}y.$$

Substituting $\epsilon = 1$ and $\epsilon = 0$, we obtain the convergences (3.2) and (3.3) respectively. □

Step 2) Derivation of the first correction term. We start by a crucial lemma which strongly relies on the time-reversibility of X.

Lemma 6. *Let* $C_1(T)$ *be the first correction term of the finite-time expansion* (3.6). *Then*

$$C_1(T) = \int_{\mathbb{R}^d}h^{(1)}(x)\mathbb{E}_x\left[e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1})\frac{\mathbb{1}_{\tau_1<T}}{\lambda_{0,A}}\right]\mu^X(x)\mathrm{d}x, \qquad (3.19)$$

where $\tau_1 \overset{d}{=} Exp\left(\lambda_{0,A}\right)$ *is independent of the Brownian motion* W.

Proof. We will first derive the explicit form of $\tilde{X}_{1,T}$. Starting from (3.5) for the equation of $\tilde{X}_{1,.}$, then applying Proposition 5 and finally taking advantage of the Hurwitz property of A (e.g. **(H-i)**), we get

$$\tilde{X}_{1,T} = \int_0^T e^{-A(T-s)}\beta(\tilde{X}_s)\mathrm{d}s = \int_0^T e^{-A_0 r}\beta(\tilde{X}_{T-r})\frac{1}{\lambda_{0,A}}\lambda_{0,A}e^{-\lambda_{0,A}r}\mathrm{d}r. \quad (3.20)$$

By the definition of $C_1(T)$, we have

$$\begin{aligned} C_1(T) &= \mathbb{E}_{\mu^X}\left[h^{(1)}\left(\tilde{X}_T\right)e^{-A_0\tau_1}\beta(\tilde{X}_{T-\tau_1})\frac{\mathbb{1}_{\tau_1<T}}{\lambda_{0,A}}\right] \\ &= \mathbb{E}_{\mu^X}\left[h^{(1)}\left(\tilde{X}_0\right)e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1})\frac{\mathbb{1}_{\tau_1<T}}{\lambda_{0,A}}\right], \end{aligned} \quad (3.21)$$

where the last equality comes from the time-reversibility of $\tilde{X}$ (valid under assumption **(H-i)**-**(H-ii)**, see Proposition 3), and the independence between τ_1 and the Brownian motion W (and thus between τ_1 and $\tilde{X}$). □

Completion of the derivation of the first order correction term. If $\varphi(x) : \mathbb{R}^d \to \mathbb{R}^d$, then recall that $\mathrm{div}\left(\varphi(x)\right) = \mathrm{Tr}\left(\nabla\varphi(x)\right)$. Using (2.9), the identity

$$\frac{\nabla\mu^X(x)}{\mu^X(x)} = -x^\top V_\infty^{-1} \quad (3.22)$$

and (3.19), we obtain after an integration by parts

$$\begin{aligned} C_1(T) &= \int_{\mathbb{R}^d} -h(x)\mathrm{div}\left(\mathbb{E}_x\left[e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1})\frac{\mathbb{1}_{\tau_1<T}}{\lambda_{0,A}}\right]\mu^X(x)\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} -h(x)\left(\left(\nabla\mu^X(x)\right)\mathbb{E}_x\left[e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1})\frac{\mathbb{1}_{\tau_1<T}}{\lambda_{0,A}}\right]\right. \\ &\qquad\left. +\mathbb{E}_x\left[\mathrm{div}\left(e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1})\right)\frac{\mathbb{1}_{\tau_1<T}}{\lambda_{0,A}}\right]\mu^X(x)\right)\mathrm{d}x \\ &\overset{(3.22)}{=} \int_{\mathbb{R}^d} h(x)\mathbb{E}_x\left[\frac{x^\top V_\infty^{-1}e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1})}{\lambda_{0,A}}\mathbb{1}_{\tau_1<T}\right. \\ &\qquad\left. -\frac{\mathrm{Tr}\left(e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A\tau_1}\right)}{\lambda_{0,A}}\mathbb{1}_{\tau_1<T}\right]\mu^X(x)\mathrm{d}x \\ &:= \int_{\mathbb{R}^d} h(x)c_1(T,x)\mu^X(x)\mathrm{d}x. \end{aligned} \quad (3.23)$$

By Lemmas 1 and 5, we see that $c_1(T,x)$ is bounded uniformly in T. We can easily apply Lebesgue dominated convergence theorem to show that

$$\begin{aligned} c_1(x) &:= \lim_{T\to+\infty} c_1(T,x) \\ &= \mathbb{E}_x\left[\frac{x^\top V_\infty^{-1}e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1})}{\lambda_{0,A}} - \frac{\mathrm{Tr}\left(e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A\tau_1}\right)}{\lambda_{0,A}}\right], \end{aligned}$$

which is exactly the announced formula for (3.9).

Step 3) Bounds on $c_1(T,x)$ and $c_1(x)$. From Equation (3.23), the above formula for c_1 and from Lemma 5, we have

$$
\begin{aligned}
&\sup_{T\geq 0}|c_1(T,x)|\vee|c_1(x)| \\
&\qquad\qquad \leq \int_0^{+\infty}\Big(c_A|x|\left\|V_\infty^{-1}\right\|e^{-\lambda_{0,A}r}|\beta|_\infty + d\,c_A^2|\beta^{(1)}|_\infty e^{-2\lambda_{0,A}r}\Big)\mathrm{d}r \\
&\qquad\qquad = \frac{c_A}{\lambda_{0,A}}|x|\left\|V_\infty^{-1}\right\||\beta|_\infty + \frac{d\,c_A^2|\beta^{(1)}|_\infty}{2\lambda_{0,A}}.
\end{aligned}
$$

The bounds on $c_1(T,x)$ and $c_1(x)$ are now proved. They are sufficient to pass to the limit in (3.23) and obtain

$$
\lim_{T\to+\infty}C_1(T) = \int_{\mathbb{R}^d}h(x)c_1(x)\mu^X(x)\mathrm{d}x. \tag{3.24}
$$

Step 4) Error estimates. In view of the representation (3.7) of the error at fixed time, we easily deduce, in view of Lemmas 3 and 4,

$$
\begin{aligned}
\sup_{T\geq 0}|\mathrm{Error}_1(T)| \leq c\,\Big(&\left(c_{(3.1)}|\beta|_\infty\right)^2|h^{(2)}|_\infty \\
&+\Big(c_{(3.1)}^2|\beta^{(1)}|_\infty|\beta|_\infty + c_{(3.1)}^3|\beta^{(2)}|_\infty|\beta|_\infty^2\Big)|h^{(1)}|_\infty\Big).
\end{aligned}
$$

Step 5) Conclusion. Take the limit of Equation (3.6) as $T\to+\infty$, in combination with (3.24) and Lemma 2: it shows that $\mathrm{Error}_1(T)$ has a limit as $T\to+\infty$, which we denote by Error_1, and this limit is bounded by $\sup_{T\geq 0}\mathrm{Error}_1(T)$. The proof of Theorem 1 is now complete. □

3.4 Proof of Theorem 2

As before, the proof is split into multiple steps.

Step 1) Preliminary results. The results presented in this step are necessary for our analysis. First observe that under **(H-iv)**, $\beta^{(2)} = \nabla\,\mathrm{vec}(\beta^{(1)})$ and $\beta^{(3)} = \nabla\,\mathrm{vec}(\beta^{(2)})$ are well defined and bounded.

Proposition 6. *The following derivatives are computed w.r.t. x, the starting point of the OU process $\tilde{X}$:*

$$
\begin{aligned}
\nabla\Big(e^{-As}\beta^{(1)}(\tilde{X}_s)e^{-At}\beta(\tilde{X}_{s+t})\Big) &= \Big(\Big(\beta^\top(\tilde{X}_{s+t})e^{-A^\top t}\Big)\otimes e^{-As}\Big)\beta^{(2)}(\tilde{X}_s)e^{-As} \\
&\quad + e^{-As}\beta^{(1)}(\tilde{X}_s)e^{-At}\beta^{(1)}(\tilde{X}_{s+t})e^{-A(s+t)}, \qquad (3.25) \\
\nabla\left(x^\top V_\infty^{-1}e^{-Aq}\beta(\tilde{X}_q)\right) &= \beta^\top(\tilde{X}_q)e^{-A^\top q}V_\infty^{-1} \\
&\quad + x^\top V_\infty^{-1}e^{-Aq}\beta^{(1)}(\tilde{X}_q)e^{-Aq}, \qquad (3.26) \\
\nabla\,\mathrm{Tr}\left(\left(e^{-2A^\top q}\right)^\top\beta^{(1)}(\tilde{X}_q)\right) &= \left(\mathrm{vec}\left(e^{-2A^\top q}\right)\right)^\top\beta^{(2)}(\tilde{X}_q)e^{-Aq}. \qquad (3.27)
\end{aligned}
$$

Proof. Note that $\partial_x \tilde{X}_{\tau_1} = e^{-A\tau_1}$ under the $\mathbb{P}_x$-measure, see (2.1). To prove Equation (3.25), we have to recall the fact that $\mathrm{vec}\,(M_1 M_2 M_3) = \left(M_3^\top \otimes M_1\right)\mathrm{vec}\,(M_2)$. We prove Equation (3.26) by a direct matrix differentiation. Lastly, to prove Equation (3.27), we recall that $\mathrm{Tr}\left(M_1^\top M_2\right) = \left(\mathrm{vec}\,(M_1)\right)^\top \mathrm{vec}\,(M_2)$. ☐

Proposition 7. *The following equality holds:*

$$\int_0^{+\infty}\int_0^v e^{-m\lambda_{0,A}r}e^{-n\lambda_{0,A}v}\mathrm{d}r\mathrm{d}v = \frac{1}{n\,(m+n)\,\lambda_{0,A}^2}.$$

Proof. We have

$$\begin{aligned}\int_0^{+\infty}\int_0^v e^{-m\lambda_{0,A}r}e^{-n\lambda_{0,A}v}\mathrm{d}r\mathrm{d}v &= \frac{1}{m\lambda_{0,A}}\int_0^{+\infty} e^{-n\lambda_{0,A}v} - e^{-(m+n)\lambda_{0,A}v}\mathrm{d}v\\ &= \frac{1}{m\lambda_{0,A}}\left(\frac{1}{n\lambda_{0,A}} - \frac{1}{(m+n)\,\lambda_{0,A}}\right)\\ &= \frac{1}{n\,(m+n)\,\lambda_{0,A}^2}.\end{aligned}$$

☐

Lemma 7. *Assume* **(H-i)**-**(H-iv)**-**(H-vi)**. *We have, for any* $\varepsilon \in [0,1]$,

$$\begin{aligned}\sup_{T\geq 0}\left|\tilde{X}^\epsilon_{3,T}\right| \leq c c^3_{(3.1)}|\beta|_\infty \Big(&\left(|\beta^{(1)}|_\infty + c_{(3.1)}|\beta^{(2)}|_\infty|\beta|_\infty\right)^2\\ &+ |\beta|_\infty\left(|\beta^{(2)}|_\infty + c_{(3.1)}|\beta^{(3)}|_\infty|\beta|_\infty\right)\Big).\end{aligned}$$

Proof. From Equation (3.18), we obtain

$$\begin{aligned}\mathrm{d}\tilde{X}^\epsilon_{3,t} = \Big[&- A\tilde{X}^\epsilon_{3,t} + \epsilon\beta^{(1)}(\tilde{X}^\epsilon_{0,t})\tilde{X}^\epsilon_{3,t} + 3\beta^{(1)}(\tilde{X}^\epsilon_{0,t})\tilde{X}^\epsilon_{2,t} + 3\partial_\epsilon\left(\beta^{(1)}(\tilde{X}^\epsilon_{0,t})\right)\tilde{X}^\epsilon_{1,t}\\ &+ 2\epsilon\partial_\epsilon\left(\beta^{(1)}(\tilde{X}^\epsilon_{0,t})\right)\tilde{X}^\epsilon_{2,t} + \epsilon\partial^2_\epsilon\left(\beta^{(1)}(\tilde{X}^\epsilon_{0,t})\right)\tilde{X}^\epsilon_{1,t}\Big]\mathrm{d}t.\end{aligned}$$

Therefore, invoking Proposition 5, Lemma 3 and 4 together with the facts that $\partial_\epsilon\left(\beta^{(1)}(\tilde{X}^\epsilon_{0,t})\right)$ has a norm bounded by $c|\beta^{(2)}|_\infty \sup_{T\geq 0}|\tilde{X}^\epsilon_{1,T}|$ and $\partial^2_\epsilon\left(\beta^{(1)}(\tilde{X}^\epsilon_{0,t})\right)$ has a norm that can be bounded by $c\Big(|\beta^{(3)}|_\infty(\sup_{T\geq 0}|\tilde{X}^\epsilon_{1,T}|)^2 + |\beta^{(2)}|_\infty\sup_{T\geq 0}|\tilde{X}^\epsilon_{2,T}|\Big)$ yields the announced result after factorization. ☐

Step 2) Derivation of the second correction term. The decomposition of $\mathbb{E}_{\mu^X}\left[h(\tilde{Y}_T)\right]$ to the second order correction term is

$$\mathbb{E}_{\mu^X}\left[h(\tilde{Y}_T)\right] = \mathbb{E}_{\mu^X}\left[h(\tilde{X}_{0,T})\right] + C_1(T) + C_{21}(T) + C_{22}(T) + \mathrm{Error}_2(T) \quad (3.28)$$

where

$$C_{21}(T) := \frac{1}{2}\mathbb{E}_{\mu^X}\left[h^{(1)}(\tilde{X}_{0,T})\tilde{X}_{2,T}\right],$$

$$C_{22}(T) := \frac{1}{2}\mathbb{E}_{\mu^X}\left[(\tilde{X}_{1,T})^\top h^{(2)}(\tilde{X}_{0,T})\tilde{X}_{1,T}\right]$$

and $\mathrm{Error}_2(T)$ as shown in Equation (3.33).

Now, we introduce 2 lemmas which rely on the time-reversibility of X.

Lemma 8. *$C_{21}(T)$ is the first part of the second correction term of our finite-time expansion* (3.28)*, and it can be represented as*

$$C_{21}(T) = \int_{\mathbb{R}^d} h^{(1)}(x)\mathbb{E}_x\left[e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A_0\tau_2}\beta(\tilde{X}_{\tau_1+\tau_2})\frac{\mathbb{1}_{\tau_1+\tau_2<T}}{\lambda_{0,A}^2}\right]\mu^X(x)\mathrm{d}x,$$

with τ_1 and τ_2 as in Theorems 1 and 2.

Proof. We take $\epsilon = 0$ for Equation (3.18) before using Proposition 5 and invoking Equation (3.20) to obtain

$$\begin{aligned}X_{2,T} &= \int_0^T\int_0^q 2e^{-A(T-q)}\beta^{(1)}(\tilde{X}_q)e^{-Au}\beta(\tilde{X}_{q-u})\mathrm{d}u\mathrm{d}q\\ &\overset{(r=T-q)}{=} \int_0^T\int_0^{T-r} 2e^{-A_0 r}\beta^{(1)}(\tilde{X}_{T-r})e^{-A_0 u}\beta(\tilde{X}_{T-r-u})\\ &\qquad\qquad \frac{\lambda_{0,A}e^{-\lambda_{0,A}r}\lambda_{0,A}e^{-\lambda_{0,A}u}}{\lambda_{0,A}^2}\mathrm{d}u\mathrm{d}r.\end{aligned}$$

From the time-reversibility of the OU process $\tilde{X}$ and the independence of τ_1 and τ_2 with the Brownian Motion W and thus with $\tilde{X}$, we have

$$\begin{aligned}C_{21}(T) &= \mathbb{E}_{\mu^X}\left[\frac{h^{(1)}(\tilde{X}_T)e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{T-\tau_1})e^{-A_0\tau_2}\beta(\tilde{X}_{T-\tau_1-\tau_2})}{\lambda_{0,A}^2}\mathbb{1}_{\tau_1+\tau_2<T}\right]\\ &= \int_{\mathbb{R}^d} h^{(1)}(x)\mathbb{E}_x\left[\frac{e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A_0\tau_2}\beta(\tilde{X}_{\tau_1+\tau_2})}{\lambda_{0,A}^2}\mathbb{1}_{\tau_1+\tau_2<T}\right]\mu^X(x)\mathrm{d}x.\end{aligned}$$

□

Lemma 9. *$C_{22}(T)$ is the second part of the second order correction term of our finite-time expansion* (3.28)*, it has the representation*

$$C_{22}(T) := \int_{\mathbb{R}^d}\mathbb{E}_x\left[\frac{\beta^\top(\tilde{X}_{\tau_1})e^{-A_0^\top\tau_1}h^{(2)}(x)e^{-A_0\tau_2}\beta(\tilde{X}_{\tau_2})}{\lambda_{0,A}^2}\mathbb{1}_{\tau_1<\tau_2<T}\right]\mu^X(x)\mathrm{d}x,$$

with τ_1 and τ_2 as in Theorems 1 and 2.

Proof. We merely proceed as in the proof of Lemma 8. Start from the definition of $C_{22}(T)$, writing the quadratic term in $\tilde{X}_{1,T}$ as a double time-integral owing to (3.20), use the symmetry in the time-integral; it readily follows that

$$C_{22}(T) = \mathbb{E}_{\mu^X}\left[\frac{\beta^\top(\tilde{X}_{T-\tau_1})e^{-A_0^\top\tau_1}h^{(2)}(\tilde{X}_T)e^{-A_0\tau_2}\beta(\tilde{X}_{T-\tau_2})}{\lambda_{0,A}^2}\mathbb{1}_{\tau_1<\tau_2<T}\right].$$

Now, we conclude by using the time-reversibility property. □

Completion of the derivation of the second order correction term. From Lemma 8, an integration by parts and Proposition 6 with Equations (3.22) and (1.4) gives

$$
\begin{aligned}
C_{21}(T) &= \int_{\mathbb{R}^d} h(x)\mathbb{E}_x\Bigg[\Bigg\{\frac{x^\top V_\infty^{-1}e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A_0\tau_2}\beta(\tilde{X}_{\tau_1+\tau_2})}{\lambda_{0,A}^2} \\
&\quad - \mathrm{Tr}\left(\frac{\left\{\left(\beta^\top(\tilde{X}_{\tau_1+\tau_2})e^{-A_0^\top\tau_2}\right)\otimes e^{-A_0\tau_1}\right\}\beta^{(2)}(\tilde{X}_{\tau_1})e^{-A\tau_1}}{\lambda_{0,A}^2}\right) \\
&\quad - \mathrm{Tr}\left(\frac{e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A_0\tau_2}\beta^{(1)}(\tilde{X}_{\tau_1+\tau_2})e^{-A(\tau_1+\tau_2)}}{\lambda_{0,A}^2}\right)\Bigg\}\mathbb{1}_{\tau_1+\tau_2<T}\Bigg]\mu^X(x)\mathrm{d}x \\
&:= \int_{\mathbb{R}^d} h(x)c_{21}(T,x)\mu^X(x)\mathrm{d}x. \qquad (3.29)
\end{aligned}
$$

Next, from Lemma 9, an integration by parts gives (after some simplifications)

$$
\begin{aligned}
C_{22}(T) &= -\sum_{i,j=1}^d \int_{\mathbb{R}^d} \partial_{x_i}h(x) \\
&\qquad \mathbb{E}_x\left[\partial_{x_j}\left(\left[e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1})\right]_i\left[e^{-A_0\tau_2}\beta(\tilde{X}_{\tau_2})\right]_j\mu^X(x)\right)\frac{\mathbb{1}_{\tau_1<\tau_2<T}}{\lambda_{0,A}^2}\right]\mathrm{d}x \\
&= -\int_{\mathbb{R}^d} h^{(1)}(x)\mathbb{E}_x\Big[\Big(e^{-A\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A_0(\tau_1+\tau_2)}\beta(\tilde{X}_{\tau_2}) \\
&\qquad -x^\top V_\infty^{-1}e^{-A_0\tau_2}\beta(\tilde{X}_{\tau_2})e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1}) \\
&\qquad + \mathrm{Tr}\left(e^{-A_0\tau_2}\beta^{(1)}(\tilde{X}_{\tau_2})e^{-A\tau_2}\right)e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1})\Big)\frac{\mathbb{1}_{\tau_1<\tau_2<T}}{\lambda_{0,A}^2}\Big]\mu^X(x)\mathrm{d}x.
\end{aligned}
$$

Applying another integration by parts yields

$$
\begin{aligned}
C_{22}(T) = \int_{\mathbb{R}^d} h(x)\mathbb{E}_x\Big(\Big\{&-x^\top V_\infty^{-1}e^{-A\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A_0(\tau_1+\tau_2)}\beta(\tilde{X}_{\tau_2}) \\
&+ \mathrm{Tr}\left(\left\{\left(\beta^\top(\tilde{X}_{\tau_2})e^{-A_0^\top(\tau_1+\tau_2)}\right)\otimes e^{-A\tau_1}\right\}\beta^{(2)}(\tilde{X}_{\tau_1})e^{-A\tau_1}\right) \\
&+ \mathrm{Tr}\left(e^{-A\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A_0(\tau_1+\tau_2)}\beta^{(1)}(\tilde{X}_{\tau_2})e^{-A\tau_2}\right) \\
&+x^\top V_\infty^{-1}e^{-A_0\tau_2}\beta(\tilde{X}_{\tau_2})x^\top V_\infty^{-1}e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1}) \\
&-x^\top V_\infty^{-1}e^{-A\tau_2}\beta^{(1)}(\tilde{X}_{\tau_2})e^{-A_0(\tau_1+\tau_2)}\beta(\tilde{X}_{\tau_1}) \\
&-\beta^\top(\tilde{X}_{\tau_2})e^{-A_0^\top\tau_2}V_\infty^{-1}e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1}) \\
&-x^\top V_\infty^{-1}e^{-A_0\tau_2}\beta(\tilde{X}_{\tau_2})\,\mathrm{Tr}\left(e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A\tau_1}\right) \\
&-x^\top V_\infty^{-1}e^{-A_0\tau_1}\beta(\tilde{X}_{\tau_1})\,\mathrm{Tr}\left(e^{-A_0\tau_2}\beta^{(1)}(\tilde{X}_{\tau_2})e^{-A\tau_2}\right) \\
&+\left(\mathrm{vec}\left(e^{-2A^\top\tau_2}\right)\right)^\top\beta^{(2)}(\tilde{X}_{\tau_2})e^{-A_0(\tau_1+\tau_2)}\beta(\tilde{X}_{\tau_1}) \\
&+\mathrm{Tr}\left(e^{-A_0\tau_2}\beta^{(1)}(\tilde{X}_{\tau_2})e^{-A\tau_2}\right)\mathrm{Tr}\left(e^{-A_0\tau_1}\beta^{(1)}(\tilde{X}_{\tau_1})e^{-A\tau_1}\right)\Big\}
\end{aligned}
$$

$$
\left.\frac{\mathbb{1}_{\tau_1<\tau_2<T}}{\lambda_{0,A}^2}\right] \mu^X(x)\mathrm{d}x
$$

$$
:= \int_{\mathbb{R}^d} h(x)c_{22}(T,x)\mu^X(x)\mathrm{d}x. \tag{3.30}
$$

So, as for $c_1(x)$ we can justify that as $T \to +\infty$, $c_{21}(T,x) \to c_{21}(x)$ and $c_{22}(T,x) \to c_{22}(x)$. Thus, we also have

$$
\lim_{T\to+\infty} C_{21}(T) = \int_{\mathbb{R}^d} h(x)c_{21}(x)\mu^X(x)\mathrm{d}x, \tag{3.31}
$$

$$
\lim_{T\to+\infty} C_{22}(T) = \int_{\mathbb{R}^d} h(x)c_{22}(x)\mu^X(x)\mathrm{d}x. \tag{3.32}
$$

The details are left for the readers.

Step 3) Bounds on $c_{21}(T,x)$ and $c_{21}(x)$. From (3.29) and Proposition 7, we have

$$
\begin{aligned}
&\sup_{T\geq 0} |c_{21}(T,x)| \vee |c_{21}(x)| \\
&\leq \int_{[0,+\infty]^2} \Big(c_A^2|x| \left\|V_\infty^{-1}\right\| e^{-\lambda_{0,A}(r+v)}|\beta|_\infty|\beta^{(1)}|_\infty \\
&\qquad + dc_A^3 e^{-\lambda_{0,A}(2r+v)}|\beta|_\infty|\beta^{(2)}|_\infty + dc_A^3 e^{-2\lambda_{0,A}(r+v)}|\beta^{(1)}|_\infty^2 \Big)\mathrm{d}v\mathrm{d}r \\
&= \frac{c_A^2\left\|V_\infty^{-1}\right\||\beta|_\infty|\beta^{(1)}|_\infty|x|}{\lambda_{0,A}^2} + \frac{dc_A^3|\beta|_\infty|\beta^{(2)}|_\infty}{2\lambda_{0,A}^2} + \frac{dc_A^3|\beta^{(1)}|_\infty^2}{4\lambda_{0,A}^2}.
\end{aligned}
$$

The bounds on $c_{21}(T,x)$ and $c_{21}(x)$ are proved.

Step 4) Bounds on $c_{22}(T,x)$ and $c_{22}(x)$. From (3.30) and Proposition 7, we easily obtain

$$
\begin{aligned}
&\sup_{T\geq 0} |c_{22}(T,x)| \vee |c_{22}(x)| \\
&\leq c_A^2|x| \left\|V_\infty^{-1}\right\| |\beta^{(1)}|_\infty|\beta|_\infty \frac{1}{3\lambda_{0,A}^2} + dc_A^3|\beta^{(2)}|_\infty|\beta|_\infty \frac{1}{4\lambda_{0,A}^2} \\
&\quad + dc_A^3|\beta^{(1)}|_\infty^2 \frac{1}{8\lambda_{0,A}^2} + c_A^2|x|^2 \left\|V_\infty^{-1}\right\|^2 |\beta|_\infty^2 \frac{1}{2\lambda_{0,A}^2} \\
&\quad + c_A^2|x| \left\|V_\infty^{-1}\right\| |\beta^{(1)}|_\infty|\beta|_\infty \frac{1}{6\lambda_{0,A}^2} + c_A^2 \left\|V_\infty^{-1}\right\| |\beta|_\infty^2 \frac{1}{2\lambda_{0,A}^2} \\
&\quad + dc_A^3|x| \left\|V_\infty^{-1}\right\| |\beta^{(1)}|_\infty|\beta|_\infty \frac{1}{3\lambda_{0,A}^2} + dc_A^3|x| \left\|V_\infty^{-1}\right\| |\beta^{(1)}|_\infty|\beta|_\infty \frac{1}{6\lambda_{0,A}^2} \\
&\quad + c_A c_{\mathrm{vec}(-2A^\top)}|\beta^{(2)}|_\infty|\beta|_\infty \frac{1}{12\lambda_{0,A}^2} + d^2c_A^4|\beta^{(1)}|_\infty^2 \frac{1}{8\lambda_{0,A}^2} \\
&= \frac{c_A|\beta^{(2)}|_\infty|\beta|_\infty}{4\lambda_{0,A}^2}\left(dc_A^2 + \frac{c_{\mathrm{vec}(-2A^\top)}}{3}\right) + \frac{dc_A^3|\beta^{(1)}|_\infty^2}{8\lambda_{0,A}^2}(dc_A+1)
\end{aligned}
$$

$$
+\frac{c_A^2 \left\|V_\infty^{-1}\right\| |\beta^{(1)}|_\infty |\beta|_\infty |x|}{2\lambda_{0,A}^2}(1+dc_A)+\frac{c_A^2 \left\|V_\infty^{-1}\right\| |\beta|_\infty^2}{2\lambda_{0,A}^2}\left(|x|^2 \left\|V_\infty^{-1}\right\|+1\right).
$$

The bounds on $c_{22}(T,x)$ and $c_{22}(x)$ are proved.

Step 5) Error estimates. By a Taylor expansion of $\epsilon \mapsto \mathbb{E}_{\mu^X}\left[h(\tilde{X}_T^\epsilon)\right]$, we directly obtain that the error up to the second order correction term is given as

$$
\begin{aligned}
\mathrm{Error}_2(T) = \frac{1}{2}\int_0^1 (1-u)^2\, \mathbb{E}_{\mu^X}\Big[\Big((\tilde{X}_{1,T}^u)^\top \otimes (\tilde{X}_{1,T}^u)^\top\Big)\, h^{(3)}(\tilde{X}_T^u)\tilde{X}_{1,T}^u \\
+3(\tilde{X}_{2,T}^u)^\top h^{(2)}(\tilde{X}_{0,T}^u)\tilde{X}_{1,T}^u + h^{(1)}(\tilde{X}_{0,T}^u)\tilde{X}_{3,T}^u\Big]\,\mathrm{d}u. \quad (3.33)
\end{aligned}
$$

Thus, in view of the above representation of the error at a fixed time T, we deduce easily the inequality (3.14) from Lemmas 3, 4 and 7.

Step 6) Conclusion. Taking the limit of Equation (3.28) as $T \to +\infty$, in combination with Equations (3.24), (3.31), (3.32) and Lemma 2, we show that $\mathrm{Error}_2(T)$ has a limit as $T \to +\infty$ which is denoted by Error_2 and the limit is bounded by $\sup_{T\geq 0}\mathrm{Error}_2(T)$. The proof of Theorem 2 is thus completed. □

4 Application to simulation algorithm, numerical results

In the sequel, we assume the previous hypotheses of Theorems 1 and 2 are in force.

4.1 Generalized acceptance-rejection algorithm

We give a simulation scheme, in a form that suits well the decompositions (3.12) and (3.15). The aim is to sample according to a probability distribution μ^+ built from a slightly non-positive unit-mass measure μ; moreover, we control the total variation distance between the two measures, in terms of the negative part of μ.

Theorem 3. *Let μ be a measure on $\mathbb{R}^d$ with mass 1 but not necessarily positive, that can be written as:*

$$
\mu(\mathrm{d}x) = \mathbb{E}\left[\xi\left(Y^x, x\right)\right]\mu_0(\mathrm{d}x),
$$

where $\mu_0(\mathrm{d}x)$ is a probability measure, the distribution of Y^x depends on x and there exists a function $H : \mathbb{R}^d \mapsto \mathbb{R}^+$ such that

$$
\left|\left(\xi\left(y,x\right)\right)^+\right| \leq H(x), \qquad \forall x, y,
$$

with $\int_{\mathbb{R}^d} H(x)\mu_0(\mathrm{d}x) < +\infty$. Moreover, assume that $0 < \int_{\mathbb{R}^d}\mathbb{E}\left[\left(\xi\left(Y^x,x\right)\right)^+\right]\mu_0(\mathrm{d}x) < +\infty$ and set

$$
\mu^+(\mathrm{d}x) := \frac{\mathbb{E}\left[\left(\xi\left(Y^x,x\right)\right)^+\right]\mu_0(\mathrm{d}x)}{\int_{\mathbb{R}^d}\mathbb{E}\left[\left(\xi\left(Y^x,x\right)\right)^+\right]\mu_0(\mathrm{d}x)}.
$$

Then Algorithm 1 produces a random variate with distribution μ^+, with an acceptance rate equal to

$$p := \frac{\int_{\mathbb{R}^d} \mathbb{E}\left[(\xi(Y^x, x))^+\right] \mu_0(\mathrm{d}x)}{\int_{\mathbb{R}^d} H(z)\mu_0(\mathrm{d}z)}$$

and an error in total variation given by

$$|\mu - \mu^+|_{\mathrm{TV}} \leq 2 \int_{\mathbb{R}^d} \mathbb{E}\left[(\xi(Y^x, x))^-\right] \mu_0(\mathrm{d}x). \tag{4.1}$$

```
repeat
    Generate X with distribution H(x)μ0(dx) / ∫H(x)μ0(dx) (on R^d)
    Generate Y with distribution Y^x for x = X
    Generate a uniform [0,1] random variate U
until UH(x) ≤ (ξ(Y,X))^+;
Return X
```

Algorithm 1: Generalized acceptance-rejection method

Proof. Denoting by $g_x(\mathrm{d}y)$ the distribution of Y^x, for any Borel set $\mathcal{B}$ we have

$$\begin{aligned}
\mathbb{P}(X\,(\textit{returned}) \in \mathcal{B}) &= \frac{1}{p}\mathbb{P}\left(X \in \mathcal{B}, UH(x) \leq (\xi(Y, X))^+\right) \\
&= \frac{1}{p}\int_{\mathcal{B}}\int \frac{(\xi(y,z))^+}{H(z)} \frac{H(z)\mu_0(\mathrm{d}z)}{\int H(x)\mu_0(\mathrm{d}x)} g_z(\mathrm{d}y) \\
&= \frac{1}{p\int H(x)\mu_0(\mathrm{d}x)} \int_{\mathcal{B}} \mathbb{E}\left[(\xi(Y^z, z))^+\right]\mu_0(\mathrm{d}z) \\
&= \frac{\int_{\mathbb{R}^d} \mathbb{E}\left[(\xi(Y^z, z))^+\right]\mu_0(\mathrm{d}x)}{p\int_{\mathbb{R}^d} H(x)\mu_0(\mathrm{d}x)} \int_{\mathcal{B}} \mu^+(\mathrm{d}z).
\end{aligned}$$

Therefore, by setting $\mathcal{B} = \mathbb{R}^d$ it is clear that the acceptance probability is p, as advertised, and that the variate produced by the algorithm has distribution μ^+. When $\xi(Y^x, x)$ is not non-negative, μ and μ^+ may be different. Then the total variation distance between μ^+ and μ is

$$\begin{aligned}
&\sup_{h:|h|_\infty \leq 1} \left| \int_{\mathbb{R}^d} h(x)\mu(\mathrm{d}x) - \int_{\mathbb{R}^d} h(x)\mu^+(\mathrm{d}x) \right| \\
&= \sup_{h:|h|_\infty \leq 1} \Bigg| \int_{\mathbb{R}^d} h(x) \\
&\qquad \frac{\mathbb{E}\left[\xi(Y^x, x)\right] \int_{\mathbb{R}^d} \mathbb{E}\left[(\xi(Y^x, x))^+\right]\mu_0(\mathrm{d}x) - \mathbb{E}\left[(\xi(Y^x, x))^+\right]}{\int_{\mathbb{R}^d} \mathbb{E}\left[(\xi(Y^x, x))^+\right]\mu_0(\mathrm{d}x)} \mu_0(\mathrm{d}x) \Bigg|
\end{aligned}$$

(since $\xi(Y^x, x) = (\xi(Y^x, x))^+ - (\xi(Y^x, x))^-$)

$$= \sup_{h:|h|_\infty \leq 1} \Bigg| \int_{\mathbb{R}^d} h(x) \Bigg(\frac{\mathbb{E}\left[(\xi(Y^x, x))^+\right]\left(\int_{\mathbb{R}^d} \mathbb{E}\left[(\xi(Y^x, x))^+\right]\mu_0(\mathrm{d}x) - 1\right)}{\int_{\mathbb{R}^d} \mathbb{E}\left[(\xi(Y^x, x))^+\right]\mu_0(\mathrm{d}x)}$$

$$- \mathbb{E}\left[(\xi(Y^x, x))^-\right]\Big)\mu_0(\mathrm{d}x)\Bigg|$$

(since $1 = \int_{\mathbb{R}^d} \mathbb{E}\left[\xi(Y^x, x)\right]\mu_0(\mathrm{d}x)$)

$$= \sup_{h:|h|_\infty \leq 1} \Bigg| \int_{\mathbb{R}^d} h(x) \Bigg(\frac{\mathbb{E}\left[(\xi(Y^x, x))^+\right] \left(\int_{\mathbb{R}^d} \mathbb{E}\left[(\xi(Y^x, x))^-\right]\mu_0(\mathrm{d}x)\right)}{\int_{\mathbb{R}^d} \mathbb{E}\left[(\xi(Y^x, x))^+\right]\mu_0(\mathrm{d}x)}$$
$$- \mathbb{E}\left[(\xi(Y^x, x))^-\right]\Bigg)\mu_0(\mathrm{d}x)\Bigg|$$
$$\leq 2\int_{\mathbb{R}^d} \mathbb{E}\left[(\xi(Y^x, x))^-\right]\mu_0(\mathrm{d}x).$$

□

4.2 Simulation algorithm for 1st and 2nd order approximations

4.2.1 First order simulation scheme

repeat
- Generate $K \overset{d}{=} Bern\left(\frac{p_{11}}{p_{11}+p_{12}}\right)$; `/* sampling of Gaussian mixture */`
- **if** $K = 1$ **then**
  - Generate $X \overset{d}{=} \mu^X$
- **else**
  - Generate $X \overset{d}{=} \mathcal{N}(0, 2V_\infty)$
- Generate $R \overset{d}{=} Exp(\lambda_{0,A})$; `/* sampling of` τ_1 `*/`
- Generate $Y \overset{d}{=} \mathcal{N}\left(e^{-AR}X, V_\infty\left(\mathrm{Id} - e^{-2A^\top R}\right)\right)$; `/* sampling of` $\tilde{X}_{\tau_1}$ `using (2.4) */`
- Generate a uniform $[0,1]$ random variate U

until $UH_1(X) \leq (\xi_1(R, Y, X))^+$;

Return X

Algorithm 2: sampling of μ^Y at the first-order accuracy

According to Theorem 1, we have

$$\mu^{Y,1}(\mathrm{d}x) = \mathbb{E}_x\left[\xi_1\left(\tau_1, \tilde{X}_{\tau_1}, x\right)\right]\mu^X(\mathrm{d}x),$$

with

$$\xi_1(r, y, x) = 1 + \frac{x^\top V_\infty^{-1} e^{-A_0 r}\beta(y)}{\lambda_{0,A}} - \frac{\mathrm{Tr}\left(e^{-A_0 r}\beta^{(1)}(y)e^{-Ar}\right)}{\lambda_{0,A}}.$$

Furthermore, if we let

$$p_{11} := 1 + \frac{|\beta^{(1)}|_\infty}{\lambda_{0,A}} d c_A^2,$$
$$p_{12} := 2^{d/2} c_A \left| x \exp\left(-\frac{1}{4} x^\top V_\infty^{-1} x\right)\right|_\infty \|V_\infty^{-1}\| \frac{|\beta|_\infty}{\lambda_{0,A}},$$

then

$$|\xi_1(r,y,x)| \leq H_1(x) := p_{11} + p_{12} 2^{-d/2} \exp\left(\frac{1}{4} x^\top V_\infty^{-1} x\right),$$
$$\frac{H_1(x)\mu^X(\mathrm{d}x)}{\int_{\mathbb{R}^d} H_1(x)\mu^X(\mathrm{d}x)} \overset{d}{=} \frac{p_{11}}{p_{11}+p_{12}} \mu^X(\mathrm{d}x) + \frac{p_{12}}{p_{11}+p_{12}} \mathcal{N}(0, 2V_\infty).$$

In this form, we obtain that the distribution to simulate in Algorithm 1-Line 2 is simply a mixture of two Gaussian distributions. As a consequence, Algorithm 2 produces a random variate with distribution

$$\left(\mu^{Y,1}\right)^+ (\mathrm{d}x) = \frac{\mathbb{E}\left[(\xi_1(\tau_1, \tilde{X}_{\tau_1}, x))^+\right] \mu^X(\mathrm{d}x)}{\int_{\mathbb{R}^d} \mathbb{E}\left[(\xi_1(\tau_1, \tilde{X}_{\tau_1}, x))^+\right] \mu^X(\mathrm{d}x)}$$

with an acceptance rate $p = \frac{\int_{\mathbb{R}^d} \mathbb{E}\left[\left(\xi_1\left(\tau_1, \tilde{X}_{\tau_1}, x\right)\right)^+\right] \mu^X(\mathrm{d}x)}{p_{11}+p_{12}}$. Recall that when $\beta \to 0$, ξ_1 converges to 1: thus, for small β, the acceptance rate p is close to 1. Starting from (4.1) and invoking Gaussian tail estimates, we can derive a sharp estimate on the error in TV between $\mu^{Y,1}$ and $\left(\mu^{Y,1}\right)^+$. This is stated as follows, the easy proof is left to the reader.

Proposition 8. *If $|\beta^{(1)}|_\infty < \frac{\lambda_{0,A}}{d c_A^2}$, then the total variation distance is controlled as*

$$\left|\mu^{Y,1} - \left(\mu^{Y,1}\right)^+\right|_{\mathrm{TV}} \leq m_{1,1} e^{-\frac{m_{1,2}}{|\beta|_\infty^2}},$$

where the positive constants $m_{1,1}$ and $m_{1,2}$ depend on the model parameters and are locally uniform when $|\beta|_\infty$ and $|\beta^{(1)}|_\infty$ tend to 0.

This shows that $\mu^{Y,1}$ and $\left(\mu^{Y,1}\right)^+$ are exponentially close to each other as $|\beta|_\infty$ and $|\beta^{(1)}|_\infty$ are small.

4.2.2 Second order simulation scheme

From Theorem 2, we have

$$\mu^{Y,2}(\mathrm{d}x) = \mathbb{E}_x\left[\xi_2(\tau_1, \tau_2, \tilde{X}_{\tau_1}, \tilde{X}_{\tau_2}, \tilde{X}_{\tau_1+\tau_2}, x)\right] \mu^X(\mathrm{d}x)$$

with an explicit function $\xi_2(\cdot)$ (derived from $c_1(x), c_{21}(x), c_{22}(x)$). Furthermore, if we let

$$p_{21} := 1 + \frac{d c_A^2 |\beta^{(1)}|_\infty}{\lambda_{0,A}} + \frac{d c_A^3 (2 + d c_A) |\beta^{(1)}|_\infty^2}{\lambda_{0,A}^2}$$

repeat

  Generate $K \overset{d}{=} Bern\left(\frac{p_{21}}{p_{21}+p_{22}}\right)$; `/* sampling of Gaussian mixture */`

  if $K=1$ **then**

    Generate $X \overset{d}{=} \mu^X$

  else

    Generate $X \overset{d}{=} \mathcal{N}\left(0, 2V_\infty\right)$

  Generate $R \overset{d}{=} Exp\left(\lambda_{0,A}\right)$; `/* sampling of` τ_1 `*/`

  Generate $V \overset{d}{=} Exp\left(\lambda_{0,A}\right)$; `/* sampling of` τ_2 `*/`

  if $R \leq V$ **then**

    Generate $Y \overset{d}{=} \mathcal{N}\left(e^{-AR}X, V_\infty\big(\mathrm{Id}-e^{-2A^\top R}\big)\right)$

    Generate $W \overset{d}{=} \mathcal{N}\left(e^{-A(V-R)}Y, V_\infty\big(\mathrm{Id}-e^{-2A^\top (V-R)}\big)\right)$

    Generate $Z \overset{d}{=} \mathcal{N}\left(e^{-AR}W, V_\infty\big(\mathrm{Id}-e^{-2A^\top R}\big)\right)$

  else

    Generate $W \overset{d}{=} \mathcal{N}\left(e^{-AV}X, V_\infty\big(\mathrm{Id}-e^{-2A^\top V}\big)\right)$

    Generate $Y \overset{d}{=} \mathcal{N}\left(e^{-A(R-V)}W, V_\infty\big(\mathrm{Id}-e^{-2A^\top (R-V)}\big)\right)$

    Generate $Z \overset{d}{=} \mathcal{N}\left(e^{-AV}Y, V_\infty\big(\mathrm{Id}-e^{-2A^\top V}\big)\right)$

  Generate a uniform $[0,1]$ random variate U

until $UH_2(X) \leq \left(\xi_2(R,V,Y,W,Z,X)\right)^+$;

Return X

Algorithm 3: sampling of μ^Y at the second-order accuracy

$$+\frac{c_A\left(2dc_A^2+c_{\mathrm{vec}(-2A^\top)}\right)|\beta|_\infty|\beta^{(2)}|_\infty}{\lambda_{0,A}^2}+\frac{c_A^2\left\|V_\infty^{-1}\right\||\beta|_\infty^2}{\lambda_{0,A}^2},$$

$$p_{22} := 2^{d/2}\left\|V_\infty^{-1}\right\|\left|x\exp\left(-\frac{1}{4}x^\top V_\infty^{-1}x\right)\right|_\infty \left(\frac{c_A|\beta|_\infty}{\lambda_{0,A}}+\frac{3c_A^2|\beta|_\infty|\beta^{(1)}|_\infty}{\lambda_{0,A}^2}+\frac{2dc_A^3|\beta|_\infty|\beta^{(1)}|_\infty}{\lambda_{0,A}^2}\right)$$
$$+\frac{2^{d/2}c_A^2\left|xx^\top\exp\left(-\frac{1}{4}x^\top V_\infty^{-1}x\right)\right|_\infty\left\|V_\infty^{-1}\right\|^2|\beta|_\infty^2}{\lambda_{0,A}^2},$$

then

$$|\xi_2(r,v,y,w,z,x)| \leq H_2(x) := p_{21}+p_{22}2^{-d/2}\exp\left(\frac{1}{4}x^\top V_\infty^{-1}x\right),$$
$$\frac{H_2(x)\mu^X(\mathrm{d}x)}{\int_{\mathbb{R}^d}H_2(x)\mu^X(\mathrm{d}x)} \overset{d}{=} \frac{p_{21}}{p_{21}+p_{22}}\mu^X(\mathrm{d}x)+\frac{p_{22}}{p_{21}+p_{22}}\mathcal{N}\left(0,2V_\infty\right).$$

Similar to the first-order simulation scheme, Algorithm 3 produces a random variate with distribution $\left(\mu^{Y,2}\right)^+(\mathrm{d}x)$ (defined in Theorem 3) with an acceptance rate close to 1 as β is small and such that the distance in TV between $\left(\mu^{Y,2}\right)^+$ and $\mu^{Y,2}$ is exponentially small as $\beta \to 0$. Details are left to the reader.

Proposition 9. *As $\beta \to 0$, we have*

$$\left|\mu^{Y,2} - \left(\mu^{Y,2}\right)^+\right|_{\mathrm{TV}} \leq m_{2,1} e^{-\frac{m_{2,2}}{|\beta|_\infty^2}},$$

where the positive constants $m_{2,1}$ and $m_{2,2}$ depend on the model parameters and are locally uniform when $|\beta|_\infty$, $|\beta^{(1)}|_\infty$ and $|\beta^{(2)}|_\infty$ tend to 0.

4.3 Numerical experiments

Two-dimensional example. In this subsection, we first conduct numerical study for $d = 2$, $\Sigma = \mathrm{Id}$, with A and β given by

$$A = A^\top = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \quad \text{and} \quad \beta\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} b\sin(x_1) \\ b\sin(x_2) \end{pmatrix},$$

where $a_{11}, a_{22}, b \in \mathbb{R}$. We compare using contour plots our approximate distribution generated with 4 millions sample points with the exact distribution for $a_{11} = 7$, $a_{22} = 7.1$ and $b = 0.1$. The density of the exact distribution[1] of Y is $\mathrm{Cst} \times \exp\left(-7x_1^2 - 7.1x_2^2 - 2b\cos(x_1) - 2b\cos(x_2)\right)$.

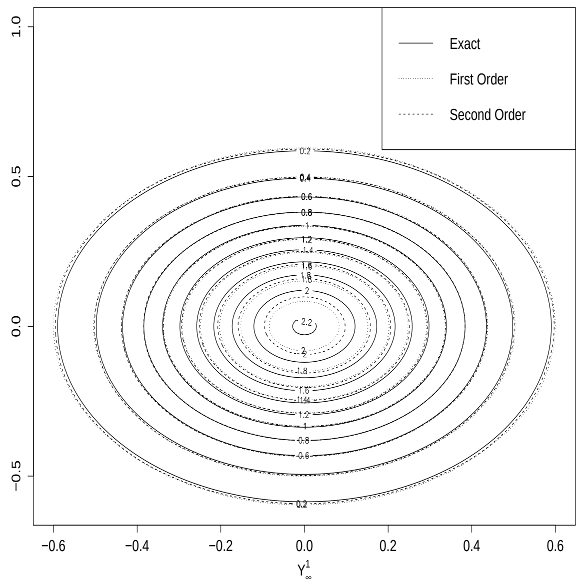

Figure 1: Density plot for the stationary distribution of $\mathrm{d}\tilde{Y}_t = \left(-A\tilde{Y}_t + \beta(\tilde{Y}_t)\right)\mathrm{d}t + \mathrm{d}W_t$, where $a_{11} = 7$, $a_{22} = 7.1$ and $b = 0.1$.

We see from Figure 1 that our approximation is accurate in approximating the exact distribution and that the second order approximation is better than

[1] Indeed the drift is of the form $-\frac{1}{2}\nabla V(x)$ so that the stationary density is $\mathrm{Cst} \times e^{-V(x)}$.

the first order approximation. Furthermore, we also note that because of the Gaussian exponential decay, the errors are larger in the center of the distributions than in the tails.

One-dimensional example. Next, we conduct numerical studies for $d = 1$, $\Sigma = 1$, $A = 1.7$ and $\beta(x) = bxe^{-x^2}$, for $b = 1, 1.6, 1.7$ and 5 with 4 millions sample points. The density of the exact distribution for this model is Cst $\times$ $\exp\left(-1.7x^2 - be^{-x^2}\right)$. Taking different values for b (thus changing $|\beta^{(1)}|_\infty$) for a fixed A serves to investigate to which extent the condition **(H-vi)** is important for the algorithm accuracy.

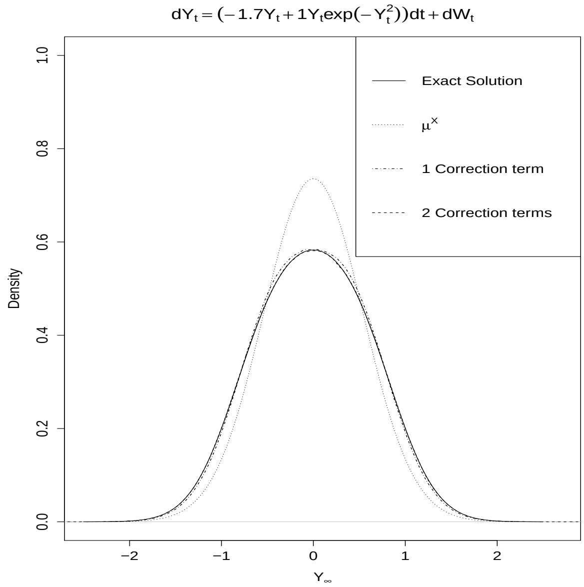

Figure 2: Density plot for the stationary distribution of $\mathrm{d}Y_t = \big(-1.7Y_t + Y_t e^{-Y_t^2}\big)\mathrm{d}t + \mathrm{d}W_t$.

In the case $b = 1$, we notice again from Figure 2 that our approximation is very accurate and that the second order approximation is superior compared to the first order.

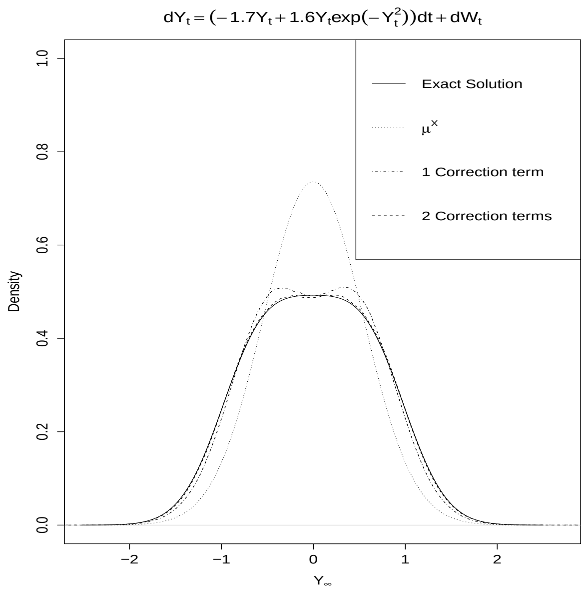

Figure 3: Density plot for the stationary distribution of $\mathrm{d}Y_t = \big(-1.7Y_t + 1.6Y_t e^{-Y_t^2}\big)\mathrm{d}t + \mathrm{d}W_t$.

Here, we compare the results for $b = 1$ and $b = 1.6$ (resp. on Figures 2 and 3), to observe that our approximation is more accurate when the ratio of $|\beta|_\infty$ against A is smaller. This is in agreement with our theoretical error analysis.

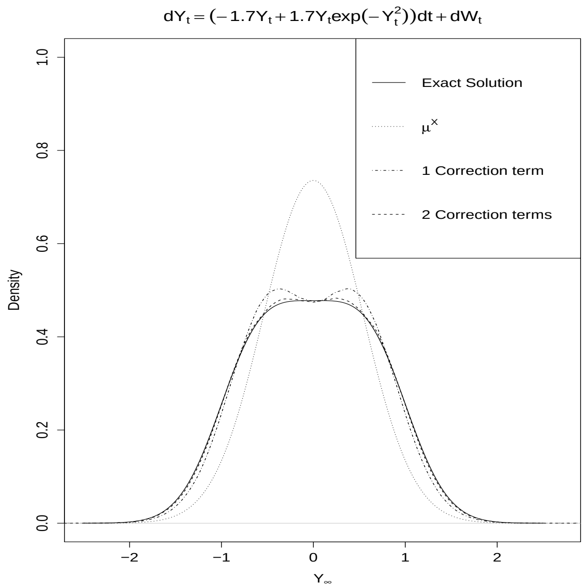

Figure 4: Density plot for the stationary distribution of $\mathrm{d}Y_t = \big(-1.7Y_t + 1.7Y_t e^{-Y_t^2}\big)\mathrm{d}t + \mathrm{d}W_t$.

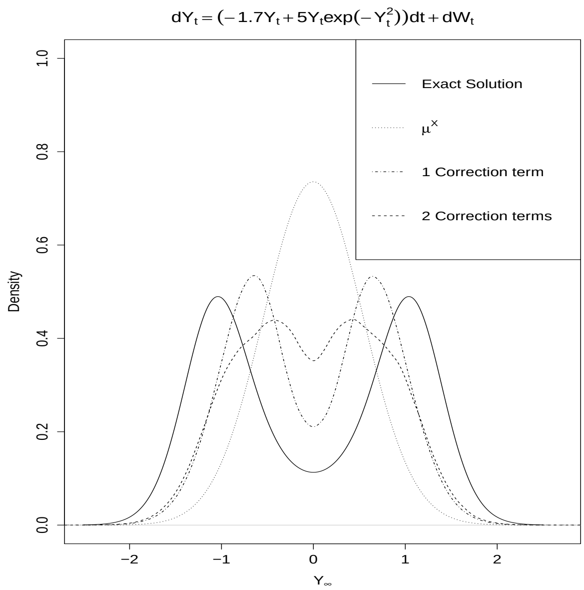

Figure 5: Density plot for the stationary distribution of $\mathrm{d}Y_t = \big(-1.7Y_t + 5Y_t e^{-Y_t^2}\big)\mathrm{d}t + \mathrm{d}W_t$.

In Figure 4 where $b = 1.7$, we notice that although $|\beta^{(1)}|_\infty$ is no longer smaller than A, our approximation can still work. However, we observe in Figure 5 ($b = 5$) that, when $|\beta^{(1)}|_\infty \gg A$, then our approximation is not accurate anymore, it only gives a rough approximation of the solution. In this case, both first and second order schemes are inaccurate; nevertheless, the second order approximation is seemingly worse, this reinforces the role of the assumptions in this analysis.

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